

Chow groups, Deligne cohomology and massless matter in F-theory

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Abstract

We propose a method to compute the exact number of charged localized massless matter states in an F-theory compactification on a Calabi-Yau 4-fold with non-trivial 3-form data. Our starting point is the description of the 3-form data via Deligne cohomology. A refined cycle map allows us to specify concrete elements therein in terms of the second Chow group of the 4-fold, i.e. rational equivalence classes of algebraic 2-cycles. We use intersection theory within the Chow ring to extract from this data a line bundle class on the curves in the base of the fibration on which charged matter is localized. The associated cohomology groups are conjectured to count the exact massless spectrum, in agreement with general patterns in Type IIB compactifications with 7-branes. We exemplify our approach by calculating the massless spectrum in an $SU(5) \times U(1)$ toy model based on an elliptic 4-fold with an extra section. The explicit evaluation of the cohomology classes is performed with the help of the *cohomCalc*-algorithm by Blumenhagen et al.

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1 Introduction

The central question of this work concerns the way how the exact charged massless spectrum is encoded in the 'gauge bundle data' of an F-theory [1, 2, 3] compactification to four dimensions. Apart from posing an exciting mathematical challenge by itself, the computation of the exact charged spectrum is of great relevance to phenomenological applications of F-theory as pioneered in [4, 5, 6, 7] and studied intensively in the recent literature. The massless gauge and matter degrees of freedom of an F-theory compactification are understood most generally by duality with M-theory. Non-abelian gauge bosons on a 7-brane along a complex codimension-one cycle on the base B_3 of the elliptic fibration Y_4 arise from the excitations of M2-branes wrapped along combinations of vanishing \mathbb{P}^1 s in the fiber; these intersect like the affine Dynkin diagram of the

associated Lie algebra and give rise, in the zero-volume limit, to an ADE-type singularity in the fibre. Throughout this paper we will work with a Calabi-Yau resolution \hat{Y}_4 of Y_4 , assuming it exists, corresponding to a non-vanishing volume of the fibre \mathbb{P}^1 s.

Similarly, massless matter arises from suitable zero mode excitations of M2-branes wrapped around certain vanishing \mathbb{P}^1 s in the elliptic fiber. Apart from so-called bulk matter, which propagates along the entire 7-branes of the F-theory compactification, extra charged M2-brane zero modes localise on complex curves C_R in the base B_3 , where one or two 7-brane loci (self-)intersect [8]. Over these curves the singularity type of the fibre enhances, i.e. one or more of the fiber \mathbb{P}^1 s split. The representation R of the gauge group G in which the associated chiral $\mathcal{N} = 1$ supermultiplets transform is encoded in the geometry of the new \mathbb{P}^1 s over the curve C_R . The precise number of chiral $\mathcal{N} = 1$ supermultiplets in representation R and \bar{R} depends on additional 'gauge bundle data'. This data is encoded, by duality with M-theory, in the 3-form potential C_3 and its field strength $G_4 \in H^{2,2}(\hat{Y}_4)$.

At the level of the chiral index of massless states, the specification of gauge flux G_4 is sufficient. This is because the chiral index of charged massless states localised on the matter curve C_R is expressed as $\int_{A_R} G_4$, where A_R is the matter surface consisting of the wrapped \mathbb{P}^1 s fibered over C_R . While in absence of a microscopic understanding of the quantum theory of M2-branes in M-theory this formula has not been derived from first principle, it has been checked by duality with heterotic string theory [4, 9, 10], Type IIB with 7-branes [11, 12, 13, 14] and M-theory [15].

The computation of the exact massless spectrum requires a much finer specification of the 3-form data than in terms of $G_4 \in H^{2,2}(\hat{Y}_4)$. A definition of the M-theory 3-form data in topologically non-trivial situations has been given in [16, 17, 18] in the language of Cheeger-Simons twisted differential characters [19]. In this paper we will use an equivalent formulation [20, 21] in terms of Deligne cohomology [22]. The relevance of either of these two formulations for F/M-theory compactifications has been stressed in various places in the recent F-theory literature including [23, 14, 24].

Deligne cohomology can be thought of as the generalisation of the Picard group of gauge equivalence classes of line bundles to a higher form gauge theory. In particular, we can think of the Deligne cohomology group $H_D^4(X, \mathbb{Z}(2))$ as the object that fits into the short exact sequence

$$0 \longrightarrow J^2(Y_4) \longrightarrow H_D^4(Y_4, \mathbb{Z}(2)) \xrightarrow{\hat{c}_2} H_{\mathbb{Z}}^{2,2}(Y_4) \longrightarrow 0. \quad (1.1)$$

The map \hat{c}_2 associates to an element in $H_D^4(Y_4, \mathbb{Z}(2))$ a 4-form cohomology class which we interpret as a G_4 -flux. The kernel of this map consists of the flat C_3 -field configurations with values in the so-called intermediate Jacobian $J^2(Y_4)$. This generalises the more familiar relation between the gauge equivalence classes of line bundles $\text{Pic}(X)$, the first Chern class c_1 assigning to each such equivalence class a curvature 2-form and the space of flat connections (or Wilson lines).

The starting point for our analysis is the observation that a convenient way to specify such 3-form data in $H_D^4(\hat{Y}_4, \mathbb{Z}(2))$ is in terms of the Chow group $\text{CH}^2(\hat{Y}_4)$, the group of rational equivalence classes of algebraic 2-cycles on Y_4 .¹ This observation is based on the existence of a well-known map from $\text{CH}^2(\hat{Y}_4)$ to $H_D^4(\hat{Y}_4, \mathbb{Z}(2))$ - the so-called refined cycle map $\hat{\gamma}$ [22, 25] -

¹Here and in the sequel by a p -cycle we mean a cycle of *complex* dimension p .

which allows us to associate to each rational equivalence class of algebraic 2-cycles a piece of 3-form data.

The relation between algebraic cycles and gauge data in F/M-theory as such is of course not new. Indeed in [11] the existence of a map from the group of algebraic 2-cycles into $H^{2,2}(\hat{Y}_4)$ - the cycle map γ - has been used to construct interesting G_4 fluxes. The intersection number in homology with the matter surfaces computes the chiral index of charged massless modes, as described above.

Our approach is to exploit the refined cycle map to specify not only a gauge flux G_4 , but a full set of 3-form data in $H_D^4(\hat{Y}_4, \mathbb{Z}(2))$ via algebraic cycles up to rational equivalence. As will be reviewed in section 2.3, rational equivalence is a refinement of the notion of homological equivalence (see e.g. [26] for details). While two homologically equivalent algebraic cycles describe the same G_4 -flux, two rationally equivalent algebraic cycles describe even the same 3-form data up to gauge equivalence and not only the same gauge flux. Therefore the intersection product in the Chow ring [27] - i.e. the intersection product up to rational (as opposed to homological) equivalence - is a legitimate operation if we are interested in preserving the full information about the 3-form data up to gauge equivalence and not merely topological information e.g. about the chiral index.

Indeed we describe a well-defined procedure based on intersection theory on the Chow ring $\text{CH}(\hat{Y}_4)$ as developed in [27] which allows us to extract from an element in $\text{CH}^2(\hat{Y}_4)$ a line bundle class on the matter curves C_R with representative bundle L_R . We further conjecture that the localised charged massless matter multiplets of an F-theory compactification are counted by $H^i(C_R, L_R \otimes \sqrt{K_{C_R}})$ with $\sqrt{K_{C_R}}$ the spin structure on C_R determined by the holomorphic embedding of the matter curve into the Calabi-Yau 4-fold in an $\mathcal{N} = 1$ supersymmetric configuration. The very fact that the massless localised charged matter multiplets are counted by cohomology groups of the form $H^i(C_R, \mathcal{L} \otimes \sqrt{K_{C_R}})$ for some line bundle \mathcal{L} is of course well-known from Type IIB models with intersecting 7-branes [28]. It was also derived in [4, 5] in more general F-theory models from the perspective of the twisted $\mathcal{N} = 1$ gauge theory on a stack of 7-branes deformed locally into an intersecting configuration. However, a priori the relation between the line bundle \mathcal{L} and the G_4 -flux of a globally defined 4-fold is not obvious. Our analysis explains how to extract the bundle data on the matter curve for general 3-form configurations, without making use of any local approximations.

The way how to extract the line bundle L_R on C_R is essentially integration along the fiber, which, as noted above, is well-defined within rational equivalence groups. It bears some resemblance with the cylinder map exploited in the context of spectral covers [20, 9, 29, 30, 14]. Note in particular that in [31] cohomology groups counting charged matter states in spectral cover models have been explicitly computed. Our approach is generally valid also in generic F-theory compactifications which cannot be described by spectral covers. In particular this includes e.g. the spectrum of charged singlet fields on matter curves away from a 7-brane with non-abelian gauge group as appearing in 4-dimensional F-theory compactifications with extra abelian gauge groups [32, 12, 33, 13, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43]. We also point out that ref. [23] approaches the counting of zero modes arising at curves of conifold singularities in M/F-theory by analysing the stable points of the superpotential and by describing the 3-form data in the language of Cheeger-Simons differential characters.

The assertion that the above line bundle L_R on C_R indeed counts the massless zero modes

is not only natural, but also fully agrees with expectations both from a Type IIB and heterotic perspective. We exemplify this for the special case of 3-form data underlying the $U(1)$ gauge flux of an F-theory model with extra abelian gauge symmetry. To demonstrate the applicability of our approach we compute the exact massless spectrum of an $SU(5) \times U(1)_X$ F-theory compactification of the type worked out in detail in [12]. The concrete 3-form data of this toy model is chosen such that its associated flux leads to three chiral generations of $SU(5)$ matter. The problem is reduced completely to the computation of a certain line bundle cohomology on a curve on the base B_3 . If B_3 is holomorphically embedded into a smooth toric variety and for suitable 3-form data, these cohomologies can be conveniently computed in concrete examples with the help of the *cohomCalg* algorithm developed by Blumenhagen et al. [44, 45, 46, 47, 48] and standard techniques from homological algebra.

The remainder of this article is organised as follows: To set the stage we recall in section 2.1 the counting of massless matter in Type IIB compactifications with 7-branes, which we are aiming to generalise to F/M-theory. In section 2.2 we review the connection between Deligne cohomology and the 3-form data up to gauge equivalence. Our approach to specify elements in the Deligne cohomology via rational equivalence classes of algebraic cycles is described in section 2.3. For convenience of the reader we also include, in section 2.4, some of the standard properties of Chow groups with focus on the well-known intersection product which we will need for our purposes. In section 3.1 we use this machinery to extract from an element of $CH^2(\hat{Y}_4)$ a line bundle equivalence class on a matter curve C_R , whose cohomology classes are naturally conjectured to count the massless charged $\mathcal{N} = 1$ chiral supermultiplets. In section 3.2 we specify this approach to 3-form data underlying a $U(1)$ gauge flux and find full agreement with the corresponding well-established formulae in Type IIB. Section 4 applies our findings to an $SU(5) \times U(1)$ F-theory toy model. After defining the model in section 4.1, we describe in general how in suitable configurations the cohomology classes defined in section 3.2 can be computed with the help of the *cohomCalg* algorithm [44, 45, 46, 47, 48] and the use of Koszul spectral sequences. This is described in section 4.2. Finally, in section 4.3 all of this is exemplified for a 3-form configuration which gives rise to three chiral generations of $SU(5) \times U(1)$ matter, plus vectorlike pairs, whose number we compute. An outlook is given in section 5. Details of the cohomology computations of section 4 and more background on the definition of Deligne cohomology are presented in the appendices.

2 Gauge data from Deligne cohomology

2.1 Review of line bundles and their cohomologies in Type IIB

We start by reviewing in this section the description of gauge data in Type IIB string theory. This well-known material will serve as a preparation for the definition of gauge in F/M-theory, which is probably less familiar to most readers.

Consider a Type IIB orientifold compactification including a set of stacks of N_i 7-branes wrapping holomorphic cycles D_i of complex dimension two of the Calabi-Yau 3-fold X_3 .² To define the internal gauge data one has to specify in general an element of the derived bounded category $D_b(X_3)$ of coherent sheaves on X_3 . A typical element of $D_b(X_3)$ corresponds to a

²Note once more that throughout this paper a p -cycle will denote a cycle of complex dimension p .

bound state of coherent sheaves localized along the complex 2-cycles D_i , where physically the existence of a non-trivial bound state is associated with a non-zero vacuum expectation value of some of the charged open string zero modes at the intersection of the branes. These correspond to the morphisms between the sheaves in the category of coherent sheaves. For a review of this picture we refer e.g. to [49].

For simplicity we focus in the sequel on special situations in which the morphisms are turned off and the individual sheaves can be described as line bundles on the respective 4-cycles D_i . In particular we assume vanishing gluing morphisms [29, 30]. In addition we focus on abelian gauge data. With these simplifying assumptions specifying the gauge data amounts to specifying an isomorphism class of line bundles L_i on D_i with two representatives of the equivalence class being equal if the two associated line bundles differ only by a gauge transformation. As preparation for the definition of gauge data in F-theory we now briefly review how to specify such an equivalence class of line bundles.

Given a line bundle L on a smooth complex projective variety X , the first Chern class $c_1(L) = \frac{1}{2\pi i} \text{tr} F$ is an integral $(1, 1)$ -form $c_1(L) \in H_{\mathbb{Z}}^{1,1}(X)$ with $H_{\mathbb{Z}}^{1,1}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})$. Isomorphism classes of holomorphic line bundles form the Picard group $\text{Pic}(X)$. Its group structure is given by the tensor product of line bundles, that is for isomorphism classes $[L], [L']$ with representatives L, L' in $\text{Pic}(X)$, $[L] \cdot [L'] = [L \otimes L']$. The inverse of a line bundle L is the *dual line bundle* $L^* = \text{Hom}(L, \mathcal{O}_X)$. Since taking the first Chern class behaves additively with respect to the tensor product, $c_1(L \otimes L') = c_1(L) + c_1(L')$, the map $L \mapsto c_1(L)$ defines a group homomorphism from the Picard group $\text{Pic}(X)$ to $H_{\mathbb{Z}}^{1,1}(X)$, which is in fact *surjective* and whose kernel is called $\text{Pic}^0(X)$. In other words, there is an exact sequence

$$0 \rightarrow \text{Pic}^0(X) \rightarrow \text{Pic}(X) \rightarrow H_{\mathbb{Z}}^{1,1}(X) \rightarrow 0. \quad (2.1)$$

The Picard group $\text{Pic}(X)$ can be identified with the first sheaf cohomology group of \mathcal{O}_X^* , the sheaf of invertible holomorphic functions on X ,³

$$H^1(X, \mathcal{O}_X^*) \cong \text{Pic}(X). \quad (2.2)$$

Now, consider the exponential exact sequence

$$0 \rightarrow \mathbb{Z}_X \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \rightarrow 0, \quad (2.3)$$

where \mathbb{Z}_X is the sheaf of integers over X and \mathcal{O}_X denotes the sheaf of holomorphic functions on X . The first map denotes multiplication with $2\pi i$ and the second map is exponentiation. Its associated long exact sequence in cohomology

$$\cdots \rightarrow H^1(X, \mathbb{Z}) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \rightarrow \cdots \quad (2.4)$$

implies that

$$\text{Pic}^0(X) = \text{Ker}(c_1: H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X, \mathbb{Z})) \quad (2.5)$$

can be identified with $H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$. Now by Hodge theory

$$H^1(X, \mathcal{O}_X) = H^{0,1}(X) \quad (2.6)$$

³See e.g. the appendix of [50] for a quick review.

and since complex conjugation simply exchanges the grading $H^{1,0} = \overline{H^{0,1}}$, there is an isomorphism of real vector spaces

$$H^1(X, \mathbb{R}) \subset H^1(X, \mathbb{C}) \rightarrow H^{0,1}(X). \quad (2.7)$$

Therefore the lattice $H^1(X, \mathbb{Z}) \subset H^1(X, \mathbb{R})$ can be seen as a lattice in $H^{0,1}(X)$. In conclusion $\text{Pic}^0(X) = H^{0,1}(X)/H^1(X, \mathbb{Z})$ is a torus, also known as the Jacobian $J^1(X)$ of X . It parametrizes, in physics language, the inequivalent Wilson lines, i.e. the flat connections.

If X is simply connected, i.e. $\pi_1(X) = 0$, the Jacobian $J^1(X) = 0$ and a line bundle is, up to gauge equivalences, uniquely specified by its first Chern class. More generally, the details of the gauge data such as the spectrum of massless states depend on finer data than merely an element in $H_{\mathbb{Z}}^{1,1}$.

On a smooth irreducible projective variety the group $\text{Pic}(X)$ is isomorphic to the group $\text{Cl}_1(X)$ of Cartier divisors modulo linear equivalence. These in turn are isomorphic, on a smooth irreducible projective variety, to the group of Weil divisors modulo linear equivalence. The latter correspond to formal linear combinations of algebraic subvarieties of codimension 1, where two divisors are equivalent if they differ by the zeros and poles of a globally defined meromorphic function on X . This group is known as the first Chow group $\text{CH}^1(X)$ of X .

A complication arises because due to the Free-Witten anomaly, the field strength of the gauge potential on a 7-brane along D_i must be quantized such that [51]

$$\frac{1}{2\pi i} F_i + \frac{1}{2} c_1(K_{D_i}) \in H_{\mathbb{Z}}^{1,1}(D_i). \quad (2.8)$$

Here and throughout this paper we assume vanishing torsion B -field. If D_i is not spin, F is therefore not the field strength of an integer quantised line bundle due to a $1/2$ shift. This will not pose any problems for us and we will, by abuse of language, still speak of a line bundle on D_i (even though it is more correctly a $\text{Spin}_{\mathbb{C}}$ bundle [51]).

To summarize, under the simplifying assumptions stated, we can think of the gauge data of a Type IIB compactification in terms of a set of divisor classes on each of the holomorphic 2-cycles D_i wrapped by the 7-branes of the compactification.

There are two types of massless open string modes - the so-called bulk modes whose wavefunction is defined on the entire 2-cycle D_i and the localized zero modes defined at the intersection curves $C_{ab} = D_a \cap D_b$ of two 2-cycles. The most general framework to describe these is in the language of Ext groups [28]. If all sheaves are given by line bundles L_i it suffices to instead switch to the more familiar language of cohomology groups. In particular, the localized massless excitations of open strings stretched between D_a and D_b are counted by

$$H^i(C_{ab}, L_{ab} \otimes \sqrt{K_{C_{ab}}}), \quad i = 0, 1. \quad (2.9)$$

Here $L_{ab} = L_a^*|_{C_{ab}} \otimes L_b|_{C_{ab}}$ and $\sqrt{K_{C_{ab}}}$ denotes the spin bundle on C_{ab} . Let us fix our conventions such that $H^0(C_{ab}, L_{ab} \otimes \sqrt{K_{C_{ab}}})$ counts the number of $\mathcal{N} = 1$ chiral multiplets in representation $(\overline{\mathbf{N}}_a, \mathbf{N}_b)$ with respect to the gauge group $U(N_a) \times U(N_b)$ on the two brane stacks; $H^1(C_{ab}, L_{ab} \otimes \sqrt{K_{C_{ab}}})$ counts $\mathcal{N} = 1$ chiral multiplets in representation $(\mathbf{N}_a, \overline{\mathbf{N}}_b)$. If supersymmetry is broken spontaneously, the mass degeneracy between the bosonic and fermionic states within this chiral multiplet is lifted and in general only a chiral Weyl fermion remains massless. The associated index is

$$\chi_{ab} = h^0(C_{ab}, L_{ab} \otimes \sqrt{K_{C_{ab}}}) - h^1(C_{ab}, L_{ab} \otimes \sqrt{K_{C_{ab}}}) = \int_{C_{ab}} c_1(L_{ab}). \quad (2.10)$$

Note that the Freed-Witten condition (2.8) ensures that L_{ab} is integer quantized on C_{ab} . While the index depends only on the cohomology class of $c_1(L_a)$ and $c_1(L_b)$, the actual cohomology groups are sensitive to the equivalence classes of L_a and L_b as elements of $\text{Pic}(D_a)$ and $\text{Pic}(D_b)$, whose respective pullbacks to C_{ab} define the line bundle $L_{ab} \in \text{Pic}(C_{ab})$.

Finally, the spin structure appearing in (2.9) is induced by the holomorphic embedding of C_{ab} into D_a (or, equivalently, into D_b), in the following sense: By adjunction

$$K_{C_{ab}} = K_{D_a}|_{C_{ab}} \otimes N_{C_{ab}/D_a}. \quad (2.11)$$

Each of the bundles $K_{D_a}|_{C_{ab}}$ and N_{C_{ab}/D_a} in turn arise as the pullback of line bundles from the ambient Calabi-Yau 3-fold X_3 . In fact,

$$K_{D_a}|_{C_{ab}} = (K_{X_3} \otimes \mathcal{O}_{X_3}(D_a))|_{C_{ab}}, \quad N_{C_{ab}/D_a} = \mathcal{O}_{X_3}(D_b)|_{C_{ab}} \quad (2.12)$$

and thus⁴

$$K_{C_{ab}} = \mathcal{M}|_{C_{ab}}, \quad \mathcal{M} = K_{X_3} \otimes \mathcal{O}_{X_3}(D_a) \otimes \mathcal{O}_{X_3}(D_b). \quad (2.13)$$

The appearing line bundles on X_3 are uniquely determined by their first Chern class since $h^1(X_3) = 0$. If $c_1(\mathcal{M})$ is even, the bundle $\mathcal{M}^{1/2}$ is well-defined, and the claim is that the spin structure appearing in (2.9) is

$$\sqrt{K_{C_{ab}}} = \mathcal{M}^{1/2}|_{C_{ab}}. \quad (2.14)$$

Even if $\mathcal{M}^{1/2}$ per se is not well-defined as a line bundle on X_3 , the Freed-Witten quantization condition will always produce suitable combinations of bundles which are integer quantized. Of special interest to us is e.g. the situation in which the line bundles L_a and L_b themselves arise by pullback of line bundles on X_3 . By slight abuse of notation let us denote these by $\mathcal{O}_{X_3}(L_a)$ and $\mathcal{O}_{X_3}(L_b)$. In this case the Freed-Witten quantization condition ensures that $c_1(\mathcal{O}_{X_3}(L_a)) + c_1(\mathcal{O}_{X_3}(L_b)) + \frac{1}{2}c_1(\mathcal{M})$ is integer on X_3 . The pullback of the appearing combination of bundles to C_{ab} then gives the argument appearing in the cohomology groups (2.9). Similar reasoning can be applied to more general situations.

2.2 Gauge data in F/M-theory via Deligne cohomology

In F-theory compactifications, the gauge data is encoded in a rather different way. Let Y_4 denote an elliptically fibered Calabi-Yau 4-fold with projection

$$\pi : Y_4 \rightarrow B_3. \quad (2.15)$$

While the generic fiber of Y_4 is a smooth elliptic curve, its topology is more complicated in higher codimension due to fiber degenerations over the discriminant locus $\Delta \subset B_3$. We will assume that a smooth resolution \hat{Y}_4 of the singular 4-fold Y_4 exists in which the singular fibers are replaced by chains of \mathbb{P}^1 s whose intersection structure reflects the gauge and matter degrees of freedom. Recent work on the explicit construction of such a 4-fold resolution in the context of 4-dimensional F-theory compactifications includes [52, 53, 54, 55, 56, 57, 10, 12, 15, 13, 58, 59, 60]

⁴For a Calabi-Yau X_3 clearly K_{X_3} is trivial, but for later reference we keep in the subsequent formulae.

and references therein, to which we refer for more details on the fiber geometry reviewed in the next few paragraphs.

In general, the discriminant Δ splits into a number of irreducible components Δ_i as well as a remaining component Δ' . Over each Δ_i the topology of the resolved fiber reproduces the affine Dynkin diagram of a Lie algebra \mathfrak{g}_i . This gives rise to vector multiplets in the adjoint representation of \mathfrak{g}_i propagating on the component Δ_i . In Type IIB language, such a discriminant component Δ_i is to be identified with the location of a corresponding stack of 7-branes even though the types of gauge algebras \mathfrak{g}_i are more general than in perturbative Type IIB compactifications. The non-abelian gauge theory on Δ_i can be described in terms very similar to the conventional treatment of the worldvolume theory on a stack of Type IIB 7-branes. Indeed the localization of the gauge degrees of freedom on Δ_i allowed the authors of [4, 5] to invoke a description in terms of a topologically twisted eight-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory. In this approach, gauge bundle data are specified essentially in the IIB language of line (or vector) bundles defined on Δ_i , reproducing the results of [28] on the cohomology groups counting massless matter at the intersection of Δ_i with other 7-branes on B_3 .

In addition, however, the discriminant Δ contains a remaining component Δ' over which the fiber acquires an I_1 -singularity corresponding to a self-intersecting \mathbb{P}^1 . This so-called I_1 -locus would correspond, in Type IIB language, to a complicated (in general non-perturbative) bound state of all *single-wrapped* 7-branes as well as the O7-plane. Crucially, in a general F-theory compactification it is not possible to identify the individual cycles on B_3 on which such single 7-branes are wrapped and a direct definition of the gauge bundle data e.g. via the approach of [4, 5] is not possible. In particular, it is a priori not clear how to encode gauge bundle data via line bundles defined globally on a 7-brane as the very location of such single 7-branes is obscured.

In higher codimension some of the fibre components can factorize further. In codimension two this happens in the fibre over curves in the base along which two components of the discriminant intersect or the same component self-intersects. In typical examples the fibre structure in codimension-two mimics the (affine) Dynkin diagram of a higher-rank algebra into which the gauge algebras associated with the two intersecting fiber components are embedded. In the F-theory limit of vanishing fibre volume, M2-branes wrapping suitable linear combinations of \mathbb{P}^1 s in the fiber give rise to massless matter in some representation R of the participating gauge algebras. Therefore, to each representation R of massless matter one can associate a complex 2-cycle A_R - the *matter surface* - given by the fibration of a certain linear combination of fiber \mathbb{P}^1 s over a *matter curve* C_R in the base. More details will be given in section 3.1.

In F-theory additional data is provided by specifying the 3-form configuration C_3 and its associated 4-form flux $G_4 \in H^{2,2}(\hat{Y}_4)$, whose existence is inferred from duality with M-theory [61, 62, 63]. The 3-form C_3 gives rise, among others, to the degrees of freedom associated with abelian gauge bosons upon dimensional reduction. The full 3-form configuration therefore encodes the gauge data of the compactification on \hat{Y}_4 .

The choice of G_4 is subject to two transversality constraints which ensure that it has 'one leg along the fiber' [63]. More precisely, G_4 flux must satisfy the cohomological relations

$$\int_{\hat{Y}_4} G_4 \cup \pi^* \omega_4 = 0 = \int_{\hat{Y}_4} G_4 \cup [Z] \cup \pi^* \omega_2 \quad (2.16)$$

for every element $\omega_4 \in H^4(B_3)$ and $\omega_2 \in H^2(B_3)$ and with $[Z] \in H^{1,1}(\hat{Y}_4)$ denoting the class of the zero section of the fibration \hat{Y}_4 . The product is the intersection product in cohomology on \hat{Y}_4 .

Similarly to the half-integer quantisation shift (2.8) for the gauge flux in 7-brane language, G_4 is in general an element of $H_{\frac{1}{2}\mathbb{Z}}^{2,2}(\hat{Y}_4)$ because it is subject to the quantisation condition [64]

$$G_4 + \frac{1}{2}c_2(\hat{Y}_4) \in H_{\mathbb{Z}}^{2,2}(\hat{Y}_4). \quad (2.17)$$

Specifying such a G_4 flux allows one to compute the chiral index of massless matter in representation R by integrating G_4 along the complex two-cycle $A_R \in Z_2(Y)$ associated to the matter representation on C_R [4, 9, 11, 10, 12, 15, 13, 23],

$$\chi_R = \int_{A_R} G_4. \quad (2.18)$$

The fact that $\frac{1}{2} \int_{A_R} c_2(\hat{Y}_4) \in \mathbb{Z}$ guarantees that this quantity is integer [55, 13, 65].

However, G_4 does not contain sufficient data to compute the number of chiral $\mathcal{N} = 1$ multiplets in representation R and \bar{R} separately. This is to be expected from the situation in Type IIB, where, as reviewed above, we need only the first Chern class $c_1(L_{ab})$ obtained from the map $c_1: \text{Pic}(C_{ab}) \rightarrow H_{\mathbb{Z}}^{1,1}(C_{ab})$ to compute the chiral index, but the actual line bundle L_{ab} to compute $h^i(C_{ab}, L_{ab} \otimes K_{C_{ab}}^{1/2})$. Therefore the information about the M-theory 3-form encoded in $G_4 \in H_{\frac{1}{2}\mathbb{Z}}^{2,2}(\hat{Y}_4)$ must be refined in the same way as giving a class $[L_{ab}] \in \text{Pic}(C_{ab})$ refines the first Chern class $c_1(L_{ab}) \in H_{\mathbb{Z}}^{1,1}(C_{ab})$. It turns out that from a mathematical standpoint natural candidates for such a refinement live in the fourth *Deligne cohomology* class of \hat{Y}_4 . Indeed the relevance of Deligne cohomology - or equivalently of the theory of Cheeger-Simons differential characters [19] - to describe gauge data in M/F-theory has been explored in several places in the literature - see [16, 17, 18, 23] and, respectively, [29, 14, 24].

For ease of presentation we ignore the half-integer shift of G_4 for the being and focus on situations in which $G_4 \in H_{\mathbb{Z}}^{2,2}(\hat{Y}_4)$. As long as we are making this simplification we can restrict ourselves to studying *integral* Deligne cohomology. We will come back to the half-integer shift in quantisation in more general setups at the end of section 2.3.2.

Before we come to a definition of Deligne cohomology, let us state some of its properties. Deligne cohomology on a smooth complex projective variety X is graded by two integers p, q specifying the Deligne cohomology group $H_D^p(X, \mathbb{Z}(q))$, but we will only be interested in the case $p = 2q$. The Deligne cohomology group $H_D^2(X, \mathbb{Z}(1))$ is isomorphic to the Picard group

$$H_D^2(X, \mathbb{Z}(1)) \cong H^1(X, \mathcal{O}_X^*) \cong \text{Pic}(X). \quad (2.19)$$

As reviewed around equ.(2.1) it therefore fits into the short exact sequence

$$0 \rightarrow J^1(X) \rightarrow H_D^2(X, \mathbb{Z}(1)) \xrightarrow{c_1} H_{\mathbb{Z}}^{1,1}(X) \rightarrow 0, \quad (2.20)$$

with

$$J^1(X) = \frac{H^1(X, \mathbb{C})}{H^{1,0}(X, \mathbb{C}) + H^1(X, \mathbb{Z})} = H^{0,1}(X, \mathbb{C})/H^1(X, \mathbb{Z}) \quad (2.21)$$

the first intermediate Jacobian.

We would now like to describe the 'gauge data' associated not with a 1-form potential, but with a $(2p - 1)$ -form potential, where the case $p = 2$ of interest to us corresponds to the M-theory 3-form potential C_3 with field strength G_4 , the 4-form flux. The general idea is to generalize (2.20) to a short exact sequence

$$0 \rightarrow J^p(X) \rightarrow H_D^{2p}(X, \mathbb{Z}(p)) \xrightarrow{\hat{c}_p} H_{\mathbb{Z}}^{p,p}(X) \rightarrow 0. \quad (2.22)$$

Here $H_{\mathbb{Z}}^{p,p}(X) = H^{2p}(X, \mathbb{Z}) \cap H^{p,p}(X, \mathbb{C})$ is the group of *Hodge cycles*. Just like the first Chern class maps the Picard group to $H_{\mathbb{Z}}^{1,1}(X)$ by assigning to a line bundle its curvature, the above sequence associates to an element of $H_D^{2p}(X, \mathbb{Z}(p))$ a corresponding field strength with values in $H_{\mathbb{Z}}^{p,p}(X)$. This is provided by the *surjective* map⁵

$$\hat{c}_{X,p}: H_D^{2p}(X, \mathbb{Z}(p)) \rightarrow H_{\mathbb{Z}}^{p,p}(X). \quad (2.23)$$

The kernel of this map is the space of flat $(2p - 1)$ -form connections and mathematically given by the *intermediate Jacobian* $J^p(X)$, defined as

$$J^p(X) = \frac{H^{2p-1}(X, \mathbb{C})}{F^p H^{2p-1}(X, \mathbb{C}) + H^{2p-1}(X, \mathbb{Z})}. \quad (2.24)$$

Here the so-called Hodge filtration $F^p H^k(X)$

$$F^p H^k(X) = \bigoplus_{p' \geq p} H^{p', k-p'} = H^{k,0} \oplus H^{k-1,1} \oplus \dots \oplus H^{p, k-p} \quad (2.25)$$

generalizes the group $H^{1,0}(X, \mathbb{C})$ appearing in (2.21).

Specializing to the case relevant for M/F-theory on a Calabi-Yau 4-fold \hat{Y}_4 , integral Deligne cohomology $H_D^4(\hat{Y}_4, \mathbb{Z}(2))$ is an extension of the group of *Hodge cycles* $H_{\mathbb{Z}}^{2,2}(\hat{Y}_4)$ by the *intermediate Jacobian* $J^2(\hat{Y}_4)$. By means of the surjective map \hat{c}_2 it is possible to pick for any 4-form flux $G_4 \in H_{\mathbb{Z}}^{2,2}(\hat{Y}_4)$ a refinement $\mathcal{A}_G \in H_D^4(\hat{Y}_4, \mathbb{Z}(2))$ such that

$$\hat{c}_2(\mathcal{A}_G) = G_4 \in H_{\mathbb{Z}}^{2,2}(\hat{Y}_4).$$

By analogy with the special case $p = 1$ we interpret an element $\mathcal{A}_G \in H_D^4(\hat{Y}_4, \mathbb{Z}(2))$ as fixing an equivalence class of gauge bundle data modulo gauge equivalence.

The crucial insight of Deligne was that an object that fits into the short exact sequence (2.22) indeed exists and can be defined formally as the hypercohomology of the *Deligne-Beilinson complex*

$$\mathbb{Z}(p)_D = 0 \rightarrow \mathbb{Z} \xrightarrow{(2\pi i)^p} \mathcal{O}_X \rightarrow \Omega_X^1 \rightarrow \dots \rightarrow \Omega_X^{p-1}. \quad (2.26)$$

Since all we need from this well-known construction for our purposes is the existence of $H_D^4(\hat{Y}_4, \mathbb{Z}(2))$ we relegate a brief account of the Deligne-Beilinson complex and its associated hypercohomology to appendix B.

Let us stress that, as encoded in (2.22), the surjective map \hat{c}_p is not an isomorphism. Specifying a 4-form flux $G_4 \in H_{\mathbb{Z}}^{2,2}(\hat{Y}_4)$ is in general by no means sufficient to uniquely determine the gauge bundle data parametrised by $H_D^{2,2}(\hat{Y}_4, \mathbb{Z}(2))$ (in just the same manner as specifying the first Chern class of a line bundle does in general not specify the equivalence class of the line bundle). For practical purposes we therefore need a method to actually define the gauge bundle data $\mathcal{A}_G \in H_D^4(\hat{Y}_4, \mathbb{Z}(2))$ more finely than merely by specifying its associated 4-form flux $G_4 = \hat{c}_2(\mathcal{A}_G)$. This is what we turn to now.

⁵When the context admits it we will simply abbreviate $\hat{c}_{X,p}$ by \hat{c}_p .

2.3 Specifying Deligne cohomology via Chow groups

Our approach to defining an element $\mathcal{A}_G \in H_D^4(\hat{Y}_4, \mathbb{Z}(2))$ is to specify a suitable equivalence class of algebraic cycles A_G and to extract from this an object \mathcal{A}_G via the so-called refined cycle map. Before describing this, let us first review the correspondence between algebraic cycles and G_4 -fluxes. Indeed, in the recent F -theory literature it was suggested to use algebraic cycles of complex dimension 2 to specify G_4 -fluxes (see [11, 10, 12, 15, 13, 66] for early examples of G_4 gauge fluxes in fully-fledged 4-folds). Recall that the group of *algebraic cycles* $Z^p(X)$ of complex codimension p on a variety X of complex dimension d is defined to be the free group generated by irreducible subvarieties of codimension p of X . An element $C \in Z^p(X)$ is a finite sum

$$C = \sum_i n_i C_i \quad (2.27)$$

of *not necessarily smooth* irreducible subvarieties C_i of X , for some integers n_i .

To understand the connection between algebraic cycles and G_4 -fluxes, recall furthermore that to any algebraic cycle in $Z^p(X)$ one can naturally associate a cocycle in $H^{2p}(X, \mathbb{Z})$ in such a way that the resulting *cycle map*

$$\gamma_{X,p}: Z^p(X) \rightarrow H_{\mathbb{Z}}^{p,p}(X) \quad (2.28)$$

is a group homomorphism, i.e. $\gamma_{X,p}(C + C') = \gamma_{X,p}(C) + \gamma_{X,p}(C')$.⁶ It suffices to define the cycle map for a possibly singular subvariety V of codimension p and extend it linearly. If V is smooth, integration over V gives, by Poincaré duality, an element in $H^{2p}(X, \mathbb{Z})$. It is harder, but equally possible to define the cycle map for singular subvarieties and to ensure that the image indeed lies in $H^{p,p}(X)$, see chapter 11 in [67]. Thus the cycle map applied to an algebraic 2-cycle $A_G \in Z^2(\hat{Y}_4)$ on a Calabi-Yau 4-fold \hat{Y}_4 yields a candidate for a G_4 -flux, since $\gamma(A_G)$ is an element of $H_{\mathbb{Z}}^{2,2}(\hat{Y}_4)$.

As we have discussed in the preceding section, to get a handle on the gauge theoretic data in F -theory, we need to have access to more than the G_4 -flux alone. Instead we need to specify an element in Deligne cohomology $\mathcal{A}_G \in H_D^4(\hat{Y}_4, \mathbb{Z}(2))$ of the Calabi-Yau 4-fold \hat{Y}_4 . Luckily, as we will review in more detail below, for an algebraic variety X it is also possible to define a *refined cycle map* (see e.g. p.123 in [25])

$$\hat{\gamma}_{X,p}: Z^p(X) \rightarrow H_D^{2p}(X, \mathbb{Z}(p)) \quad (2.29)$$

to Deligne cohomology with the property that the cycle map $\gamma_{X,p}: Z^p(X) \rightarrow H_{\mathbb{Z}}^{p,p}(X)$ factors as $\gamma_{X,p} = \hat{c}_{X,p} \circ \hat{\gamma}_{X,p}$, where $\hat{c}_{X,p}: H_D^{2p}(X, \mathbb{Z}(p)) \rightarrow H_{\mathbb{Z}}^{p,p}(X)$ is the map defined in equation (2.23).

The refined cycle map $\hat{\gamma}_{X,p}$ allows us to access the refined gauge theoretic information contained in Deligne cohomology from algebraic geometry. That is, whenever we are able to specify an algebraic cycle $A_G \in Z^2(\hat{Y}_4)$ within the 4-fold \hat{Y}_4 , we can map it to an element $\mathcal{A}_G \in H_D^4(\hat{Y}_4, \mathbb{Z}(2))$. This map need in general not be surjective and therefore it may not give access to all possible configurations of gauge data; nonetheless we can at least specify examples of gauge configurations in this manner.

However, specifying explicit algebraic cycles introduces a huge amount of redundancy as two different algebraic cycles may map to the same element in $\mathcal{A}_G \in H_D^4(\hat{Y}_4, \mathbb{Z}(2))$ and therefore

⁶When the context is clear, we will drop the subscripts on the cycle map and denote it by γ_p or simply γ .

describe the same gauge configuration up to gauge equivalence. We will at least partially reduce this redundancy by specifying not an algebraic cycle, but rather a suitable equivalence class of algebraic cycles and define a map from this equivalence class to $H_D^{2p}(X, \mathbb{Z}(p))$. The equivalence relation is supposed to at least partially correspond to the gauge equivalence between gauge data. For the special case $p = 1$, the correct equivalence relation between algebraic codimension 1-cycles which captures gauge equivalence is known to be given by rational equivalence. Even more strongly, the first Chow group $\text{CH}^1(X)$ of algebraic codimension 1-cycles modulo rational equivalence is isomorphic to $H_D^2(X, \mathbb{Z}(1)) = \text{Pic}(X)$. For $p = 2$ an isomorphism between the analogous second Chow group $\text{CH}^2(X)$ of rationally equivalent codimension 2-cycles and $H_D^4(X, \mathbb{Z}(2))$ is unfortunately not known to be given. Nevertheless our approach will be to define elements of $H_D^4(X, \mathbb{Z}(2))$ by specifying an element of the group $\text{CH}^2(X)$. The refined cycle map (2.29) indeed maps two rationally equivalent algebraic cycles to the same element in $H_D^4(X, \mathbb{Z}(2))$ and therefore descends to a refined cycle map

$$\hat{\gamma}_{X,p}: \text{CH}^p(X) \rightarrow H_D^{2p}(X, \mathbb{Z}(p)). \quad (2.30)$$

Let us repeat this important statement: We can 'move' within the rational equivalence class of algebraic cycles without changing the associated gauge data (up to gauge equivalence). In particular, all operations such as forming intersections which are well-defined as operations on $\text{CH}(X)$ are suitable manipulations which do not change the gauge data. Indeed, the intersection product on $\text{CH}(X)$ will allow us to extract in a natural way the gauge data relevant for the counting of massless charged zero modes.

For the reader's convenience we will now give a brief overview of some of the relevant mathematical notions. For more details on this in principle well-known material see [26] and [27].

2.3.1 Chow groups

The Chow group $\text{CH}(X)$ of a d -dimensional smooth projective variety X has properties similar to both the cohomology and homology groups of a topological space. Its elements are algebraic cycles subject to a certain equivalence relation. Working with plain algebraic cycles turns out to be inconvenient. Just as in homology it is desirable to impose an equivalence relation on algebraic cycles, so that they can be "moved" within the containing algebraic variety.

The kernel of the cycle map $\gamma_{X,p}$ defined in (2.28) consists of algebraic cycles that are homologous to zero. We are interested in an equivalence relation imposed on algebraic cycles $C, C' \in Z^p(X)$ that is finer than declaring them to be equivalent $C \sim C'$ if their image under the cycle map coincides $\gamma_{X,p}(C) = \gamma_{X,p}(C')$, i.e. if they are homologous. Intuitively speaking complex geometry is rigid and thus algebraic cycles should not merely be homologous to be considered the same.

A good notion of equivalence for algebraic cycles is *rational equivalence*. It is a straightforward generalization of the notion of *linear equivalence* for Weil divisors. While there is a more geometric definition of rational equivalence available, let us first give one that makes the connection to linear equivalence obvious.

Given an invertible function $r \in K(W)^*$ on a $(k+1)$ -dimensional subvariety W of X , it is

possible to associate to it a k -cycle

$$[\operatorname{div}(r)] = \sum \operatorname{ord}_V(r)[V], \quad (2.31)$$

where the sum runs over all codimension-one subvarieties V of W and $\operatorname{ord}_V(r)$ is the *order* of r in V . For a function $f \in \mathcal{O}_X$ on an *affine* variety X the divisor of f is easy to grasp. Recall that by Krull's Hauptidealsatz the isolated components of f are all of codimension one and so the sum

$$[\operatorname{div}(f)] = \sum_V \operatorname{ord}_V(f)[V]$$

runs over a finite number of irreducible codimension one subvarieties V of X and $\operatorname{ord}_V(f)$ is simply the order of vanishing at V . This construction can be generalized beyond affine varieties.

A k -cycle $A \in Z^k(X)$ is called *rationally equivalent* to zero, $A \sim 0$, if it can be written as a finite sum $A = \sum [\operatorname{div}(r_i)]$, where $r_i \in K(W_i)^*$ are invertible rational functions in some $(k+1)$ -dimensional subvarieties W_i of X . The k -cycles rationally equivalent to zero form a subgroup $\operatorname{Rat}_k(X)$.⁷ The *Chow group* $\operatorname{CH}_k(X)$ is the group of rational equivalence classes

$$\operatorname{CH}_k(X) = Z_k(X) / \operatorname{Rat}_k(X). \quad (2.32)$$

It can be graded by either dimension or codimension. The grading by codimension will be denoted by a superscript $\operatorname{CH}^p(X)$, the grading by dimension by a subscript $\operatorname{CH}_m(X)$.

An equivalent ([27], 1.6, [26], 1.1.2) and more geometric definition of rational equivalence is as follows: Two algebraic cycles $C, C' \in Z(X)$ are rationally equivalent if there is a rationally parameterized family of cycles interpolating between them, in other words if there is a cycle on $\mathbb{P}^1 \times X$ whose restriction to two fibers $\{t\} \times X$ and $\{t'\} \times X$ are C and C' . We picture the rational equivalence between a hyperbola and the union of two lines in \mathbb{CP}^2 in Figure 1. This intuition can be made precise by defining in analogy to cohomology a “boundary map” $\partial_X: Z(X \times \mathbb{P}^1) \rightarrow Z(X)$: Let V be an irreducible subvariety of $X \times \mathbb{P}^1$, set $\partial_X(V) = 0$, if the projection $\pi: V \rightarrow \mathbb{P}^1$ is not dominant⁸ and set $\partial_X(V) = V_0 - V_\infty$ otherwise, where $V_p = \pi^{-1}(p) \subset X \times \{p\} \cong X$ are the fibers over points p in \mathbb{P}^1 . Denote the subgroup of $Z(X)$ generated by cycles of the form $V_0 - V_\infty$ by $\operatorname{Rat}(X)$ and the subgroup generated by codimension p cycles by $\operatorname{Rat}^p(X)$.

The *Chow group* $\operatorname{CH}(X)$ can then equivalently be defined to be the group of rational equivalence classes

$$\operatorname{CH}(X) = Z(X) / \operatorname{Rat}(X) = \operatorname{Coker}(\partial_X: Z(X \times \mathbb{P}^1) \rightarrow Z(X)). \quad (2.33)$$

With these definitions in place it can be demonstrated that the cycle map $\gamma_{X,p}: Z^p(X) \rightarrow H^{2p}(X, \mathbb{C})$ descends to a map $\operatorname{CH}^p(X) \rightarrow H^{2p}(X, \mathbb{C})$ and rational equivalence classes of algebraic cycles that are homologous to zero, i.e. in the kernel of $\gamma_{X,p}$, form a subgroup $\operatorname{CH}_{\operatorname{hom}}^p(X) \subset \operatorname{CH}^p(X)$.

2.3.2 Connection between Deligne Cohomology and Chow Groups

As we have seen above, the first Chow group of $\operatorname{CH}^1(X)$ of a smooth projective variety X and the Picard group $\operatorname{Pic}(X)$ of X are isomorphic. Moreover the cycle map $\gamma_1: \operatorname{CH}^1(X) \rightarrow H_{\mathbb{Z}}^{1,1}(X)$

⁷If $A \sim 0$ and $B \sim 0$, then also $A + B \sim 0$. Since $[\operatorname{div}(r^{-1})] = -[\operatorname{div}(r)]$, $A \sim 0$ implies $-A \sim 0$.

⁸Recall that a rational map is dominant if there exists an equivalent map whose image is dense in its codomain.

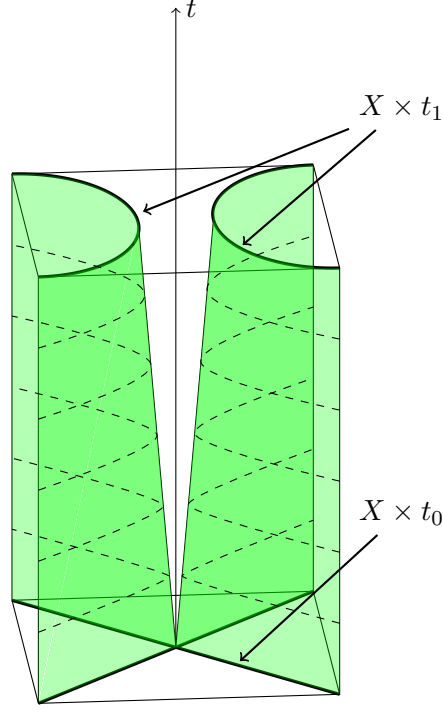


Figure 1: Rational equivalence between the union of two lines in \mathbb{CP}^2 and a hyperbola. This picture is inspired by [26].

factors through this isomorphism via the map $c_1: \text{Pic}(X) \rightarrow H_{\mathbb{Z}}^{1,1}(X)$ that associates to each isomorphism class of line bundles the first Chern class. The kernel $\text{CH}_{\text{hom}}^1(X)$ of the cycle map, i.e. the homologically trivial cycles, are mapped isomorphically to the Jacobian $J^1(X)$ of X via the Abel-Jacobi map. To summarize one has the following commuting diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{CH}_{\text{hom}}^1(X) & \longrightarrow & \text{CH}^1(X) & \xrightarrow{\gamma_1} & H_{\text{alg}}^{1,1}(X) \longrightarrow 0 \\
& & \downarrow AJ & & \downarrow & & \downarrow \\
0 & \longrightarrow & J^1(X) & \longrightarrow & \text{Pic}(X) & \xrightarrow{c_1} & H_{\mathbb{Z}}^{1,1}(X) \longrightarrow 0
\end{array}$$

Now in section 2.2 we asserted that the lower horizontal sequence $0 \rightarrow J^1(X) \rightarrow \text{Pic}(X) \rightarrow H_{\mathbb{Z}}^{1,1}(X) \rightarrow 0$ is a special case of the short exact sequence

$$0 \rightarrow J^p(X) \rightarrow H_D^{2p}(X, \mathbb{Z}(p)) \xrightarrow{\hat{\gamma}_p} H_{\mathbb{Z}}^{p,p}(X) \rightarrow 0. \quad (2.34)$$

It is therefore natural to ask whether there is an analogue of the whole diagram. We already mentioned that there is a cycle map $\gamma_p: \text{CH}^p(X) \rightarrow H_{\mathbb{Z}}^{p,p}(X)$ and it is also possible to define a generalized Abel-Jacobi map $AJ: \text{CH}_{\text{hom}}^p(X) \rightarrow J^p(X)$. So what we are looking for is a refinement $\hat{\gamma}_p: \text{CH}^p(X) \rightarrow H_D^{2p}(X, \mathbb{Z}(p))$ of the cycle map to Deligne cohomology such that the following diagram commutes:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{CH}_{\mathrm{hom}}^p(X) & \longrightarrow & \mathrm{CH}^p(X) & \xrightarrow{\gamma_p} & H_{\mathrm{alg}}^{p,p}(X) \longrightarrow 0 \\
& & \downarrow AJ & & \downarrow \hat{\gamma}_p & & \downarrow \\
0 & \longrightarrow & J^p(X) & \longrightarrow & H_D^{2p}(X, \mathbb{Z}(p)) & \xrightarrow{\hat{c}_p} & H_{\mathbb{Z}}^{p,p}(X) \longrightarrow 0
\end{array}$$

Such a refined cycle map indeed exists (see [22] §7). For the subsequent analysis, however, only its existence is of relevance and in particular the fact that rationally equivalent algebraic cycles map to the same Deligne cohomology class under $\hat{\gamma}_p$.

The existence of the refined cycle map $\hat{\gamma}_p$ for $p = 2$ therefore allows us to define an element of the Deligne cohomology group $H_D^4(\hat{Y}_4, \mathbb{Z}(2))$ on our Calabi-Yau 4-fold \hat{Y}_4 by specifying a Chow class $\alpha_G \in \mathrm{CH}^2(\hat{Y}_4)$. Thus, given a complex 2-cycle $A_G \in Z^2(\hat{Y}_4)$ whose rational equivalence class is α_G , the object $\hat{\gamma}_2(\alpha_G) \in H_D^4(\hat{Y}_4, \mathbb{Z}(2))$ defines a certain type of F/M-theory gauge data in the sense described in section 2.2. The G_4 flux associated with this gauge data is the element

$$G_4 = \gamma_2(A_G) = \hat{c}_2 \circ \hat{\gamma}_2(\alpha_G) \in H_{\mathbb{Z}}^{2,2}(\hat{Y}_4). \quad (2.35)$$

We stress, however, that the specification of α_G via A_G contains much more information than merely the G_4 -flux. This construction is the direct generalization of the specification of an equivalence class of line bundles as an element of $\mathrm{Pic}(X)$ by specifying a divisor class $\mathrm{CH}^1(X)$. While $\mathrm{Pic}(X)$ and $\mathrm{CH}^1(X)$ are isomorphic, $\mathrm{CH}^2(X)$ and $H_D^4(X, \mathbb{Z}(2))$ need in general not be isomorphic.

In practice this is not a serious drawback. Practically all G_4 gauge fluxes defined in the recent F -theory literature are in fact in the image of the cycle map $\gamma: \mathrm{CH}^2(\hat{Y}_4) \rightarrow H_{\mathbb{Z}}^{2,2}(X)$. Whenever this is the case the refined cycle map $\gamma: \mathrm{CH}^2(\hat{Y}_4) \rightarrow H_D^4(X, \mathbb{Z}(2))$ defines a compatible Deligne cohomology class. Moreover the Hodge conjecture states that if we allow algebraic cycles with rational coefficients, the cycle map will in fact be surjective. On the other hand counter examples for the case of integral coefficients are known. Since \hat{c}_2 is surjective, showing surjectivity of the refined cycle map for rational coefficients is indeed a one-million-dollar problem. The refined cycle map will also not be injective in general, that is distinct equivalence classes of algebraic cycles might map to the same Deligne cohomology class. This just means that we might not detect all equivalent gauge data in the Chow group. What is important for us is that since the refined cycle map is well-defined we will get the same Deligne cohomology class irrespective of the representative algebraic cycle we pick.

To reiterate, by the refined cycle map to every rational equivalence class of algebraic 2-cycles there exists an equivalence class of gauge data as an element in $H_D^4(\hat{Y}_4, \mathbb{Z}(2))$, and manipulations of representative cycles A_G which do not change their rational equivalence class α_G do not change the associated element in $H_D^4(\hat{Y}_4, \mathbb{Z}(2))$.

So far we have ignored the fact that G_4 is really an element of $H_{\frac{1}{2}\mathbb{Z}}^{2,2}(\hat{Y}_4)$ due to the quantization condition (2.17). This shift is taken into account by considering the Deligne cohomology group $H_D^4(\hat{Y}_4, \frac{1}{2}\mathbb{Z}(2))$ with half-integer coefficients. Similarly, via the refined cycle map one can specify an element therein via an element in the second Chow group with suitably half-integer as opposed to integer coefficients. For the associated flux $G_4 = \gamma(\alpha_G)$ to be quantised according to (2.17), the Chow group element must satisfy the condition $\gamma(\alpha_G) + \frac{1}{2}c_2(\hat{Y}_4) \in H_{\mathbb{Z}}^{2,2}(\hat{Y}_4)$. Note that the second Chern class of the tangent bundle to \hat{Y}_4 defines an element of $\mathrm{CH}^2(\hat{Y}_4)$

[26], which by abuse of notation we also denote by $c_2(\hat{Y}_4)$. Then the qunatisation condition is fulfilled if we pick $\alpha_G + \frac{1}{2}c_2(\hat{Y}_4)$ to be an element of the integer second Chow group. With the understanding that the coefficients of the Chow and Deligne cohomology groups must be chosen appropriately we will not make the half-integer shift explicit in what follows and continue to talk about $H_D^4(\hat{Y}_4, \mathbb{Z}(2))$.

2.4 Properties of Chow groups

In this section we describe operations on Chow groups with the understanding that these operations could also be directly expressed in Deligne cohomology as a result of the existence of the refined cycle map. The advantage of specifying the gauge data via Chow groups is that Chow groups are, at least in principle, suitable for explicit computation and accessible to methods of algebraic geometry. This will turn out useful when it comes to explicitly extracting the data needed to compute the charged massless matter spectrum of an F-theory compactification. As we will see, we will have to make heavy use of intersection theory within $\text{CH}(X)$. In this section we review some of the key elements of this well-developed subject [27, 26] which will be of direct relevance for section 3. The reader familiar with this material may wish to directly jump to section 3.

2.4.1 Functoriality

In order to work with Chow groups it is necessary to understand their behaviour under morphisms between varieties $f: X \rightarrow Y$. Again it is helpful to compare the situation to cohomology and homology. Homology is a covariant functor from topological spaces to abelian groups, in other words there is a pushforward map

$$f_*: H_m(X) \rightarrow H_m(Y) \quad (2.36)$$

associated to any morphism $f: X \rightarrow Y$. We would expect that by analogy there is a pushforward map $f_*: \text{CH}_m(X) \rightarrow \text{CH}_m(Y)$ preserving dimension. If f has *relative dimension* e , then f_* maps $\text{CH}^p(X) \rightarrow \text{CH}^{p-e}(Y)$.

Similarly cohomology is a contravariant functor, i.e. there exists a pullback map

$$f^*: H^m(Y) \rightarrow H^m(X) \quad (2.37)$$

associated to any morphism $f: X \rightarrow Y$. If we think of Chow groups as analogous to cohomology groups, we would think that there is a pullback map

$$f^*: \text{CH}^p(Y) \rightarrow \text{CH}^p(X) \quad (2.38)$$

preserving codimension. It turns out that both expectations are justified, with some restrictions, as we will see shortly.

The pushforward between Chow groups $f_*: \text{CH}_p(X) \rightarrow \text{CH}_p(Y)$ is defined if $f: X \rightarrow Y$ is a proper⁹ morphism between algebraic varieties: Let V be a subvariety of X , then $f(V)$ is

⁹Since we will be only concerned with complex algebraic varieties, *proper* can be understood here in the sense of the complex topology on X and Y , i.e. the map $\pi: X \rightarrow Y$ is proper if the preimage of a compact set is compact. In algebraic geometry a morphism $\pi: X \rightarrow Y$ is called proper if it is universally closed and separated.

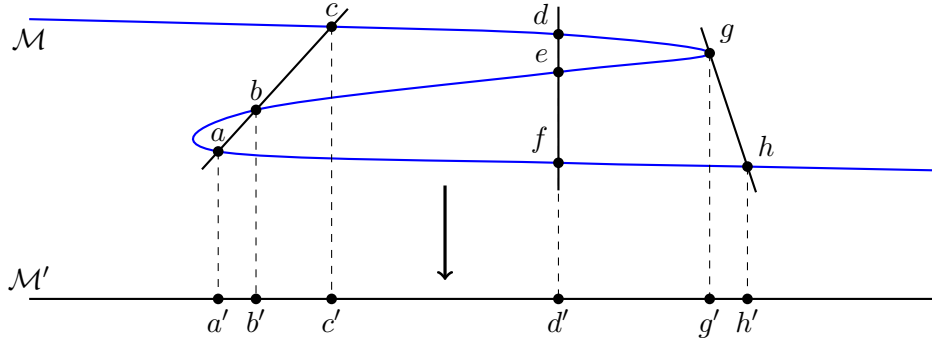


Figure 2: If $a + b + c \sim d + e + f \sim 2g + h$, then the pushforwards are equivalent cycles, i.e. $a' + b' + c' \sim 3d' \sim 2g' + h'$. This picture is based on [26].

a subvariety of Y of dimension $\dim(f(V)) \leq \dim(V)$. A guess would be that the cycle class of V in $\text{CH}(X)$ should be sent to the cycle class of $f(V)$ in $\text{CH}(Y)$. However this would not respect rational equivalence as it does not account for multiplicities in multi-sheeted covers for example.

Observe that as long as $\dim(V) = \dim(f(V))$ the map $f|_V: V \rightarrow f(V)$ is *generically finite* in the sense that the extension of function fields $K(V)/K(f(V))$ is of finite degree n . Geometrically this means that generically there are n points in the fibre over a point in the base $f(V)$. The geometric interpretation suggests taking the degree of the covering into account. Define $f_*(V) = 0$ if the dimension of $f(V)$ is strictly smaller than the dimension of V and $f_*V = nf(V)$ if $f|_V$ has degree n . This definition is extended linearly to all cycles on X by defining

$$f_*(\sum m_i V_i) = \sum m_i f_* V_i. \quad (2.39)$$

It turns out that this definition is compatible with rational equivalence ([27], Chapter 1) so we get a pushforward map

$$f_*: \text{CH}_m(X) \rightarrow \text{CH}_m(Y) \quad (2.40)$$

which is additive,

$$f_*(\alpha + \beta) = f_*(\alpha) + f_*(\beta). \quad (2.41)$$

We give a picture of this situation in Figure 2.

A good pullback map $f^*: \text{CH}(Y) \rightarrow \text{CH}(X)$ defined on cycles has to preserve algebraic equivalence and should be geometric in the sense that if, for a subvariety V of Y of codimension p , $f^{-1}(V)$ is *generically reduced* of codimension p , then the pullback of the algebraic cycle $[V]$ should be the class of the inverse image $f^{-1}(V)$: $f^*[V] = [f^{-1}(V)]$. This requirement turns out to determine f^* uniquely, at least as long as $f: X \rightarrow Y$ is a *generically separable* map between smooth algebraic varieties. Indeed, if $f: X \rightarrow Y$ is such a map, any cycle $\alpha \in \text{CH}^p(Y)$ can be represented by an algebraic cycle $A = \sum_i n_i A_i \in Z^p(Y)$, such that $f^{-1}(A_i)$ is generically reduced of codimension p for all i and the class $\sum n_i [f^{-1}(A_i)]$ in $\text{CH}^p(X)$ is independent of the choice of A .

Define the pullback $f^*(\alpha) = \sum_i n_i [f^{-1}(A_i)]$. It is then possible to show that the pullback is additive, i.e.

$$f^*(\alpha + \beta) = f^*(\alpha) + f^*(\beta). \quad (2.42)$$

2.4.2 Intersection product

One of the main motivations for working with Chow groups is to study the intersection of irreducible subvarieties A, B in X . It should be defined in such a way that it is a refinement of the *cup product* $\cup: H^i(X) \otimes H^j(X) \rightarrow H^{i+j}(X)$ in cohomology. In other words the intersection product should equip $\mathrm{CH}^p(X)$ with a ring structure and the cycle map $\gamma_X: \mathrm{CH}^p(X) \rightarrow H^{2p}(X)$ should be a ring homomorphism, i.e.

$$\gamma_X(\alpha \cdot \beta) = \gamma_X(\alpha) \cup \gamma_X(\beta) \quad (2.43)$$

for cycles $\alpha, \beta \in \mathrm{CH}(X)$.

There are different approaches to defining such an intersection product. What follows is a brief overview of the approach presented in Fulton's book on the subject [27]. The general idea is to mimic the definition of the cup-product in cohomology. Recall (see for example [68], Chapter IV) that it can be defined to be induced by the composition

$$C^\bullet(X) \times C^\bullet(X) \rightarrow C^\bullet(X \times X) \xrightarrow{\Delta^*} C^\bullet(X) \quad (2.44)$$

on cochain complexes $C^\bullet(X)$, where the first morphism is the Künneth map and the second is the pullback induced by the diagonal map $\Delta: X \rightarrow X \times X$. The role of the Künneth map is taken on by the exterior product $\times: Z_k(X) \otimes Z_l(Y) \rightarrow Z_{k+l}(X \times Y)$ of algebraic cycles that maps $A = \sum n_i A_i \in Z_k(X)$ and $B = \sum m_j B_j \in Z_l(Y)$ to $A \times B = \sum n_i m_j (A_i \times B_j) \in Z_{k+l}(X \times Y)$, where $A_i \times B_j$ is the Cartesian product of the irreducible subvarieties A_i and B_j . This product descends to a well defined exterior product $\times: \mathrm{CH}_k(X) \otimes \mathrm{CH}_l(Y) \rightarrow \mathrm{CH}_{k+l}(X \times Y)$, see [27] Chapter 1. The definition of an analogue of pullback to the diagonal $C^\bullet(X \times X) \rightarrow C^\bullet(X)$ will occupy us for the rest of this section. Notice that if X is a smooth algebraic variety the diagonal embedding $\Delta: X \rightarrow X \times X$ is a regular embedding. So what we are looking for is to define a morphism $f^*: \mathrm{CH}(Y) \rightarrow \mathrm{CH}(X)$ for any *regular embedding* $f: X \rightarrow Y$.

We will construct such a *Gysin homomorphism* $f^*: \mathrm{CH}(Y) \rightarrow \mathrm{CH}(X)$ in several steps, beginning with the case of a vector bundle $\pi: E \rightarrow X$ of rank r on X . The flat pullback map $\pi^*: \mathrm{CH}_{k-r}(X) \rightarrow \mathrm{CH}_k(E)$ is an isomorphism for all k ([27], Chapter 3). So if $s: X \rightarrow E$ is the zero section of E , in particular $\pi \circ s = \mathrm{id}_X$, then it is possible to define the *Gysin homomorphism*

$$s^*: \mathrm{CH}_k(E) \rightarrow \mathrm{CH}_{k-r}(X),$$

where s^* is the inverse of the pullback map π^*

$$s^*(\alpha) = (\pi^*)^{-1}(\alpha). \quad (2.45)$$

A closed embedding $i: X \rightarrow Y$ of varieties is a *regular embedding of codimension d* if the ideal sheaf \mathcal{I} defining X as a subvariety of Y is locally generated by a regular sequence of length d .¹⁰ Geometrically this means that X is a *local complete intersection* in Y . Relevant examples include the diagonal embedding $\delta: X \rightarrow X \times X$ if X is smooth and for any morphism $f: X \rightarrow Y$ to a non-singular variety Y the induced regular graph morphism $\gamma_f: X \rightarrow X \times Y$ given by $x \mapsto (x, f(x))$. If $i: X \rightarrow Y$ is a regular embedding of codimension d , the conormal

¹⁰A sequence $a_1, \dots, a_d \in A$ in a ring A is called regular if a_i is a non-zero divisor in $A/(a_1, \dots, a_{i-1})$

sheaf $\mathcal{I}/\mathcal{I}^2$ is a locally free sheaf of rank d and the *normal bundle* $N_X Y$ to X in Y is the vector bundle whose sheaf of sections is dual to $\mathcal{I}/\mathcal{I}^2$.

The normal cone $C_X Y$ of a closed subscheme X in Y is the spectrum of the graded ideal¹¹ $\sum_{n \geq 0} \mathcal{I}^n / \mathcal{I}^{n+1}$. If $i: X \rightarrow Y$ is a regular embedding, then the normal cone $C_X Y$ is naturally isomorphic to the normal bundle $N_X Y$. Intuitively the normal cone $C_X Y$ is a substitute for the notion of “tubular neighbourhood” from differential geometry.

In particular we can associate to any algebraic cycle on Y an algebraic cycle on $C_X Y$

$$\sigma: Z_k(Y) \rightarrow Z_k(C_X Y) \quad (2.46)$$

by the formula

$$\sigma[V] = [C_{V \cap X} V] \quad (2.47)$$

for any k -dimensional subvariety V of Y and to all k -cycles by linearity ([27], Chapter 5.2). This is well defined since $C_{V \cap X} V$ is a subvariety of pure dimension k ([27] loc. cit.).

The *specialization homomorphism* σ descends to the level of Chow groups ([27], loc. cit.):

$$\sigma: \mathrm{CH}_k(Y) \rightarrow \mathrm{CH}_k(C_X Y). \quad (2.48)$$

Let $i: X \rightarrow Y$ be a regular embedding of codimension d , with normal bundle $N = N_X Y$. Then the *Gysin homomorphism* $i^*: \mathrm{CH}_k(Y) \rightarrow \mathrm{CH}_{k-d}(X)$ is defined to be the composition $i^* = s_{N_X Y}^* \circ \sigma$,

$$\mathrm{CH}_k(Y) \xrightarrow{\sigma} \mathrm{CH}_k(N_X Y) \xrightarrow{s_N^*} \mathrm{CH}_{k-d}(X),$$

where $s_N: X \rightarrow N_X Y$ denotes the zero section of the normal bundle, s_N^* is the induced isomorphism on Chow groups and σ is the specialization homomorphism defined above.

The *intersection product* is then the composition of the exterior product with the Gysin homomorphism Δ^* induced by the diagonal embedding $\Delta: X \rightarrow X \times X$,

$$\mathrm{CH}_k(X) \otimes \mathrm{CH}_l(X) \xrightarrow{\times} \mathrm{CH}_{k+l}(X \times X) \xrightarrow{\Delta^*} \mathrm{CH}_{k+l-n}(X).$$

If α and β are cycles in $\mathrm{CH}_k(X)$ and $\mathrm{CH}_l(X)$ respectively, their intersection product in $\mathrm{CH}_{k+l}(X)$ will be denoted by $\alpha \cdot \beta$.

This notion of intersection product generalizes as follows. Suppose that $f: X \rightarrow Y$ is a morphism to a smooth variety Y of dimension n . Then the graph morphism $\gamma_f: X \rightarrow X \times Y$ given by $x \mapsto (x, f(x))$ is a regular embedding of codimension n . Notice that if f is the identity morphism, the graph morphism is just the diagonal embedding above. Define the intersection product by the composition of the exterior product with the Gysin morphism γ_f^* ,

$$\mathrm{CH}_k(X) \otimes \mathrm{CH}_l(Y) \xrightarrow{\times} \mathrm{CH}_{k+l}(X \times Y) \xrightarrow{\gamma_f^*} \mathrm{CH}_{k+l-n}(X), \quad (2.49)$$

and denote the product of two cycles α and β by

$$\alpha \cdot_f \beta. \quad (2.50)$$

¹¹Take for example the ideal $I = (xy)$ in $R = \mathbb{C}[x, y]$, $V = \mathrm{Spec}(R/I)$ is the union of the coordinate axes. The normal cone is $V \times \mathbb{A}^1$.

Since we will make use of these intersection products in the sequel, we state some of their properties (for proofs see Chapter 8 of [27]). For $\alpha = [X]$, denote $[X] \cdot_f \beta = f^* \beta$. This definition generalizes the Gysin morphism defined above.

The intersection product is associative, i.e. for $X \xrightarrow{f} Y \xrightarrow{g} Z$, with Y and Z non-singular varieties, we have

$$x \cdot_f (y \cdot_g z) = (x \cdot_f y) \cdot_{gf} z, \quad (2.51)$$

with $x \in \text{CH}(X)$, $y \in \text{CH}(Y)$ and $z \in \text{CH}(Z)$. In particular, for $y = [Y]$, we have

$$x \cdot_f g^* z = x \cdot_{gf} z, \quad (2.52)$$

since $x \cdot_f [Y] = x$. Finally, we will need the *projection formula*

$$f_*(x \cdot_{gf} z) = f_*(x) \cdot_g z. \quad (2.53)$$

In particular, if we are given a fibre square

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{p} & X \\ \downarrow q & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array} \quad (2.54)$$

and algebraic cycles $\alpha \in \text{CH}(X \times_Z Y)$ and $\beta \in \text{CH}(Z)$, then

$$q_*(\alpha \cdot_{fp} \beta) = q_*(\alpha \cdot_{gq} \beta) = q_*(\alpha) \cdot_g \beta. \quad (2.55)$$

3 Cohomology formulae for F-theory compactifications

3.1 A general cohomology formula

Let us recap the main results of the previous sections: To define the gauge bundle data of an F-theory compactification on a resolved Calabi-Yau 4-fold \hat{Y}_4 we propose to specify an element

$$\alpha_G \in \text{CH}^2(\hat{Y}_4) \quad (3.1)$$

of the rational equivalence class of complex-dimension 2-cycles on \hat{Y}_4 . With the help of the refined cycle map from $\text{CH}^2(\hat{Y}_4)$ to $H_D^4(\hat{Y}_4, \mathbb{Z}(2))$, this yields a specific type of 3-form data up to gauge transformations. A representative of the equivalence class α_G in turn is specified by an explicit 2-cycle

$$A_G \in Z^2(\hat{Y}_4) \quad (3.2)$$

whose associated Dolbeault cohomology class

$$G_4 \equiv \gamma(A_G) \in H_{\mathbb{Z}}^{(2,2)}(\hat{Y}_4) \quad (3.3)$$

is typically referred to as G_4 -flux in the F-theory literature.

Our main interest is in the fibre structure over the curves C_R in the base B_3 where massless matter in representation R of some gauge group is localised. As briefly reviewed at the beginning

of section 2.2, such matter curves correspond to loci where two components of the discriminant of the elliptic fibration intersect, or where a discriminant component self-intersects. The latter includes in particular configurations without any non-abelian gauge groups, where an essentially local approach to the gauge bundle data is not applicable. Indeed an important advantage of our approach is that it provides a way how to extract the massless matter from the gauge bundle data also in such situations.

To each matter representation R one can associate a so-called matter surface given by a complex 2-cycle

$$A_R = \sum_i n_{R,i} A_R^i \in Z^2(\hat{Y}_4), \quad (3.4)$$

where each A_R^i is given by a \mathbb{P}^1 -fibration over the curve C_R .¹² The support of A_R is the union of the support of the individual A_R^i ,

$$|A_R| = \bigcup_i |A_R^i|. \quad (3.5)$$

It is fibred over the matter curve in the base via

$$\pi_R : |A_R| \rightarrow C_R. \quad (3.6)$$

We will denote by $\alpha_R \in \text{CH}^2(\hat{Y}_4)$ the rational equivalence class with representative A_R .

In the sequel it will be useful to view A_R as a cycle on the restriction of the elliptic fibration to the curve C_R with projection

$$\pi|_{C_R} : \hat{Y}_4|_{C_R} \rightarrow C_R. \quad (3.7)$$

The preimage of a generic point on C_R with respect to $\pi|_{C_R}$ is given by the union of a number \mathbb{P}^1 s with multiplicities. Note that strictly speaking $\hat{Y}_4|_{C_R}$ is singular. Nonetheless we can view the Chow class of A_R as an element of $\text{CH}_2(\hat{Y}_4|_{C_R})$.

This structure defines in a natural way a gauge equivalence class of line bundles on the matter curve C_R . Let

$$\iota_R : \hat{Y}_4|_{C_R} \rightarrow \hat{Y}_4 \quad (3.8)$$

denote the embedding of $\hat{Y}_4|_{C_R}$ into \hat{Y}_4 . If we view α_R as an element of $\text{CH}_2(\hat{Y}_4|_{C_R})$, we are precisely in the situation described around (2.49), with the role of the map $f : X \rightarrow Y$ played by $\iota_R : \hat{Y}_4|_{C_R} \rightarrow \hat{Y}_4$. In particular we can form the intersection product

$$\alpha_R \cdot_{\iota_R} \alpha_G \in \text{CH}_0(\hat{Y}_4|_{C_R}). \quad (3.9)$$

Its push-forward to the matter curve C_R defines an element

$$\pi|_{C_R*}(\alpha_R \cdot_{\iota_R} \alpha_G) \in \text{CH}_0(C_R). \quad (3.10)$$

¹²More precisely, to each of the $\dim(R)$ weights β^k , $k = 1, \dots, \dim R$, of the representation R one associates a different matter surface $A_{R,k}$. Since nothing depends on which $A_{R,k}$ is used to compute the matter multiplicities, we simply make a convenient choice and suppress the index k in the sequel.

At this stage we must recall that α_G must be chosen as an element in the Chow ring with half-integer coefficients such that $\alpha_G + \frac{1}{2}c_2(\hat{Y}_4)$ is an integer Chow cycle class. This raises the question how (3.10) is quantised. For all concrete examples studied so far, $\frac{1}{2} \int_{A_R} c_2(\hat{Y}_4) \in \mathbb{Z}$ [55, 13, 65]. One therefore would hope that $c_2(\hat{Y}_4)$ viewed as a Chow class is such that $\frac{1}{2}\alpha_R \cdot \iota_R c_2(\hat{Y}_4)$ is integer for every matter surface. Whenever this is the case also (3.9) and (3.10) are integral. For $c_2(\hat{Y}_4)$ of the specific form described in [55, 65] considerations similar to the ones in the next section indicate that for suitably quantized α_R (3.9) and (3.10) are integral. Obviously none of those complications arise if G_4 is an integral cycle defined by an integral Chow cycle $\alpha_G \in \text{CH}^2(\hat{Y}_4)$.

The question of the integrality of (3.9) and (3.10) is important because the *integer* Chow group $\text{CH}_0(C_R) \equiv \text{CH}^1(C_R)$ is isomorphic to $\text{Pic}(C_R)$. Therefore (3.10) specifies an equivalence class of line bundles on C_R for suitably quantized α_R . Let us denote by $A_{R,G} \in Z_0(C_R)$ a representative of $\pi|_{C_R*}(\alpha_R \cdot \iota_R \alpha_G)$, which we are always free to choose. Then

$$L_{G,R} = \mathcal{O}_{C_R}(A_{R,G}) \quad (3.11)$$

is a line bundle on C_R of degree

$$\deg(L_{G,R}) = \int_{\hat{Y}_4} \gamma(A_R) \cup \gamma(A_G). \quad (3.12)$$

Let us stress that this procedure yields a line bundle on the matter curve C_R , but in general not on a higher-dimensional subvariety of the base B_3 . This, however, is sufficient to compute the localized massless matter states. In view of the expression (2.9) counting localized charge matter zero modes in Type IIB compactifications, it is natural to identify $L_{G,R}$ with the line bundle on the matter curve appearing in the generalization of (2.9) to F-theory. Our conjecture is therefore the massless matter in representation R localized on C_R is in 1-1 correspondence with the cohomology groups

$$H^i(C_R, L_{G,R} \otimes \sqrt{K_{C_R}}), \quad i = 0, 1. \quad (3.13)$$

More precisely, if the compactification preserves $\mathcal{N} = 1$ supersymmetry, $\dim H^i(C_R, L_{G,R}^G \otimes K_{C_R}^{1/2})$ counts the number $\mathcal{N} = 1$ chiral multiplets in representation R for $i = 0$ and those in representation \bar{R} for $i = 1$. As in Type IIB the spin structure $\sqrt{K_{C_R}}$ must be picked in agreement with the holomorphic embedding of the matter curve, e.g. as a complete intersection, into B_3 .

Note that the chiral index associated with the cohomology groups (3.13),

$$\chi_{R,G} = \deg(L_{G,R}) = \int_{\hat{Y}_4} \gamma(A_R) \cup \gamma(A_G), \quad (3.14)$$

is - since by construction $\gamma(A_G) = G_4$ - simply the usual expression $\int_{A_R} G_4$ that follows by duality with the heterotic string [69], with Type IIB [11, 12, 13] and via M-theory anomaly cancellation [33].

It is important to appreciate that the intersection product (3.9) underlying the definition of the line bundle $L_{G,R}$ via (3.10) is stable with respect to rational equivalence on \hat{Y}_4 . This important property was the reason for using the somewhat technical formalism described in

section 2.4. This means that the result of (3.10) does not depend on the concrete representative $A_G \in \text{CH}^2(\hat{Y}_4)$ as long as its rational equivalence class is given by α_G . This is to be contrasted with the formula (3.14) for the chiral index, which is invariant under deformations of A_G which do not change its cohomology class $\gamma(A_G)$ in $H_{\mathbb{Z}}^{2,2}(\hat{Y}_4)$. Expression (3.10) contains therefore, in general, much more information. In particular it is in general important to know how to perform intersection products in concrete examples within the Chow ring of the elliptic fibration \hat{Y}_4 .

To conclude this subsection, let us comment on a special class of gauge bundle data which, though non-generic, is of some practical relevance. Namely, suppose the Chow class α_G descends by pullback from a Chow class in a higher-dimensional ambient space of the compactification manifold \hat{Y}_4 . Indeed the elliptically fibred 4-fold \hat{Y}_4 can often be described as a hypersurface or complete intersection with a higher-dimensional variety. Consider for example the situation in which \hat{Y}_4 is embedded into a smooth 5-fold X_5 via an inclusion¹³

$$j : \hat{Y}_4 \rightarrow X_5. \quad (3.15)$$

Suppose furthermore we are interested in a special class of bundle data of the form

$$\alpha_G = j^* \tilde{\alpha}_G, \quad \tilde{\alpha}_G \in \text{CH}^2(X_5). \quad (3.16)$$

Then with the help of (2.52) the intersection product (3.9) becomes

$$\alpha_R \cdot_{\iota_R} \alpha_G = \alpha_R \cdot_{\iota_R} j^* \tilde{\alpha}_G = \alpha_R \cdot_{j \cdot \iota_R} \tilde{\alpha}_G \in \text{CH}_0(\hat{Y}_4|_{C_R}). \quad (3.17)$$

In particular, (3.17) is stable under deformations of $\tilde{\alpha}_G$ which respect rational equivalence on X_5 . This observation is of practical relevance because in many situations the Chow ring of the space X_5 may be far better under control than that of its hypersurface \hat{Y}_4 . For example, if X_5 is a smooth toric variety, then ([70], 5.2)

$$\text{CH}^*(X_5) \simeq H^*(X_5, \mathbb{Z}) \quad (3.18)$$

and in evaluating the intersection we can exploit *cohomological* relations on X_5 for $\tilde{\alpha}_G$. We stress, however, that the class of gauge bundle data of the type (3.16) is a very special subset of the general possibilities, albeit one with many applications in the recent F-theory literature.

3.2 Application to $U(1)$ gauge data

To exemplify the definition of the line bundles $L_{G,R}$ and the computation of the matter multiplicities we now specify the 4-fold \hat{Y}_4 a little further. We are interested in an F-theory compactification which exhibits, possibly in addition to some non-abelian gauge group, one or several abelian gauge group factors. The presence of non-Cartan $U(1)$ gauge group factors is tied to the existence of rational sections of the fibration [2, 3] so that the elliptic fibration \hat{Y}_4 must exhibit a Mordell-Weil group with a non-trivial finite part.¹⁴ To each abelian gauge group factor $U(1)_A$

¹³More generally, \hat{Y}_4 could be embedded also as a complete intersection into a higher dimensional ambient space.

¹⁴The construction of 4-dimensional F-theory compactifications with abelian gauge group factors via rational sections has recently received a lot of attention [32, 12, 33, 34, 35, 13, 36, 37, 38, 39, 40, 41, 42, 43]. For earlier work on multi-section fibrations and abelian gauge groups in 6-dimensional compactifications of F-theory see [71, 72, 73, 74].

one can associate a rational equivalence class

$$w_A \in \text{CH}^1(\hat{Y}_4) \quad (3.19)$$

whose associated cohomology class $\gamma(w_A) \in H_{\mathbb{Z}}^{1,1}(\hat{Y}_4)$ is determined via the Shioda map [75] from the rational sections of the elliptic fibration such that

$$\int_{\hat{Y}_4} \gamma(w_A) \cup [Z] \cup \pi^* \omega_4 = 0 = \int_{\hat{Y}_4} \gamma(w_A) \cup \pi^* \omega_6 \quad \forall \omega_4 \in H^4(B_3), \quad \omega_6 \in H^6(B_3) \quad (3.20)$$

with $[Z] \in H^{1,1}(\hat{Y}_4)$ denoting the class of the zero section of the fibration \hat{Y}_4 .

The physical significance of this transversality condition is that it guarantees that the Kaluza-Klein expansion $C_3 = A_A \wedge \tilde{w}_A + \dots$ (with \tilde{w}_A a representative of the class $\gamma(w_A)$) of the M-theory 3-form C_3 gives rise to a 1-form A_A in the external dimensions which is identified with the $U(1)_A$ gauge potential. Let us denote by

$$W_A = \sum_j m_{A,j} W_A^j \in Z_1(\hat{Y}_4) \quad (3.21)$$

an explicit cycle representative of the Chow class w_A such that W_A^j is an algebraic cycle in \hat{Y}_4 . This suffices to define a certain type of $U(1)_A$ gauge data as follows: Consider a complex 2-cycle $F \in Z_2(B_3)$ with associated Chow class $f \in \text{CH}^1(B_3)$. Then the object

$$\alpha_{F,A} = w_A \cdot_{\pi} f \in \text{CH}^2(\hat{Y}_4) \quad (3.22)$$

- again in the notation introduced (2.50) with the role of $f : X \rightarrow Y$ taken by $\pi : \hat{Y}_4 \rightarrow B_3$ - specifies a ' $U(1)_A$ bundle' whose associated 4-form class

$$G_4^A = \pi^* \gamma(f) \cup \gamma(w_A) \in H^{2,2}(\hat{Y}_4) \quad (3.23)$$

is of the form of the $U(1)_A$ -flux as introduced in [32, 11, 12, 33]. The definition of the line bundle on C_R via (3.10) is now facilitated by the projection formula (2.55), which allows us to write

$$\pi|_{C_{R*}}(\alpha_R \cdot_{\iota_R} \alpha_{F,A}) = \pi|_{C_{R*}}(\alpha_R \cdot_{\iota_R} (w_A \cdot_{\pi} f)) = \pi|_{C_{R*}}(\alpha_R \cdot_{\iota_R} w_A) \cdot_{\iota_R|_{B_3}} f. \quad (3.24)$$

Here $\iota_R|_{B_3}$ denotes the restriction of the embedding ι_R defined in (3.8) to the base B_3 of the fibration.

This can be further evaluated in the physically interesting situation in which \hat{Y}_4 is embedded, as in (3.15), into a smooth toric ambient space X_5 . Indeed in many relevant examples, the Chow class w_A associated with the Shioda map has the property of being a pullback from X_5 ,

$$w_A = j^* \tilde{w}_A. \quad (3.25)$$

With the help of (3.17) we have

$$\pi|_{C_{R*}}(\alpha_R \cdot_{\iota_R} w_A) = \pi|_{C_{R*}}(\alpha_R \cdot_{j\iota_R} \tilde{w}_A). \quad (3.26)$$

The intersection in brackets is essentially 'integration along the fibre' and since w_A arises by pullback from a toric ambient space X_5 , we are allowed to perform the intersection up to

homological equivalence on this ambient space. We are therefore precisely in the situation described in great detail in [11, 12]. The intersection in the fibre yields here $q_A(R)$ points over the curve C_R . Therefore

$$\pi|_{C_{R*}}(\alpha_R \cdot \iota_R w_A) = \pi|_{C_{R*}}(\alpha_R \cdot j_{\iota_R} \tilde{w}_A) = q_A(R) [C_R], \in \text{CH}_1(C_R), \quad (3.27)$$

where by $[C_R]$ we denote the Chow class of the curve C_R viewed as an element of $\text{CH}_1(C_R)$. The numerical prefactor $q_A(R) \in \mathbb{Z}$, which, as we said, results from the multiplicities of the intersection in the fibre, is physically interpreted as the $U(1)_A$ charge of the matter representation R [11, 12]. The remaining intersection with f must in general be performed within rational equivalence on B_3 , and it is here where the use of homological intersection theory in general comes to an end. Let us pick an explicit representative $A_{R,A} \in Z_0(C_R)$ of the Chow class $[C_R] \cdot \iota_{R|B} f$. It defines a line bundle $\mathcal{O}_{C_R}(A_{R,A})$. Then the massless matter states correspond to the cohomology classes

$$H^i(C_R, L_{R,A} \otimes \sqrt{K_{C_R}}), \quad L_{R,A} = [\mathcal{O}_{C_R}(A_{R,A})]^{\otimes q_A(R)}. \quad (3.28)$$

In geometries with a well-defined Sen limit, this agrees with the expression for the massless matter states in Type IIB orientifolds.

4 A 3-generation $SU(5) \times U(1)_X$ example

4.1 The general setup

To illustrate the presented technology we compute the massless charged spectrum of an F-theory compactification with $SU(5) \times U(1)_X$ gauge group. The simplest such compactification corresponds to what was called in [32] a $U(1)$ restricted Tate model, which describes a compactification Calabi-Yau 4-fold Y_4 given by a specific type of elliptic fibration with Mordell-Weil group of rank one over a base B_3 . The fibration is described as the hypersurface equation

$$P_T = \{y^2 s + a_1 x y z s + a_3 y z^3 = x^3 s^2 + a_2 x^2 z^2 s + a_4 x z^4\}. \quad (4.1)$$

Here $[x : y : z : s]$ denote homogeneous coordinates on the toric fibre ambient space subject to the scaling relations $(x, y, z, s) \simeq (\lambda^2 \mu^{-1} x, \lambda^3 \mu^{-1} y, \lambda^1 z, \mu s)$. The Tate polynomials a_i are sections of $K_{B_3}^{-i}$. For

$$a_2 = a_{2,1} w, \quad a_3 = a_{3,2} w^2, \quad a_4 = a_{4,3} w^3 \quad (4.2)$$

with generic $a_1, a_{2,1}, a_{3,2}, a_{4,3}$ the fibration exhibits an $SU(5)$ singularity in the fibre over the base surface

$$\{p \in B_3 : w = 0\}, \quad w \in H^0(\mathcal{O}_{B_3}(D_{\text{GUT}})). \quad (4.3)$$

We will use the resolution of this singularity as worked out in detail in [12]. Restricting to one of the existing six inequivalent triangulations, the resolved 4-fold is described by the hypersurface

$$\begin{aligned} & y^2 s e_3 e_4 + a_1 x y z s + a_{3,2} y z^3 e_0^2 e_1 e_4 \\ & = x^3 s^2 e_1 e_2^2 e_3 + a_{2,1} x^2 z^2 s e_0 e_1 e_2 + a_{4,3} x z^4 e_0^3 e_1^2 e_2 e_4. \end{aligned} \quad (4.4)$$

The resolution has introduced four resolution divisors $E_i : \{e_i = 0\}$, $i = 1, \dots, 4$, and e_0 is the proper transform of w . The scaling relations of the homogeneous coordinates can be found in table 1 of [12], to which we refer for more details of the geometry.

In addition to the holomorphic section $Z : \{z = 0\}$ the fibration exhibits a rational section $S : \{s = 0\}$. Let us denote their rational equivalence classes again by Z and S , viewed as elements of $\text{CH}^1(\hat{Y}_4)$, and similarly for E_i . By slight abuse of notation we also denote the rational equivalence class associated with the anti-canonical divisor by \bar{K}_{B_3} . Then the object [12]

$$w_X = 5(S - Z - \bar{K}_{B_3}) + \sum_i l_i E_i \in \text{CH}^1(\hat{Y}_4), \quad l_i = (2, 4, 6, 3), \quad (4.5)$$

defines, via the cycle map, an element $\gamma(w_X) \in H^{1,1}(\hat{Y}_4)$ such that expansion of the M-theory 3-form C_3 in terms of a representative of $\gamma(w_X)$ gives rise to the gauge potential A_X associated with a $U(1)_X$ gauge group.

The base B_3 contains four matter curves supporting charged matter in representations **10**₋₁, **5**₃, **5**₂ and **1**₅ plus the respective conjugates, where the subscripts denote the $U(1)_X$ charges. In agreement with their physical interpretation as Standard Model matter and, respectively, Higgs fields in an $SU(5)$ GUT model, we will denote the field **5**₃ as **5**_m and **5**₂ as **5**_H. The matter curves are given as complete intersections on B_3 ,

$$C_R = \{p \in B_3, U_R(p) = V_R(p) = 0\} \quad (4.6)$$

with

$$U_R \in H^0(B_3, \mathcal{O}_{B_3}(D_R^a)), \quad V_R \in H^0(B_3, \mathcal{O}_{B_3}(D_R^b)). \quad (4.7)$$

Specifically,

$$\begin{aligned} C_{\mathbf{10}} : \{w = 0\} \cap \{a_1 = 0\}, \quad C_{\bar{\mathbf{5}}_m} : \{w = 0\} \cap \{a_{3,2} = 0\}, \\ C_{\mathbf{5}_H} : \{w = 0\} \cap \{a_1 a_{4,3} - a_{2,1} a_{3,2} = 0\}, \quad C_{\mathbf{1}} : \{a_{3,2} = 0\} \cap \{a_{4,3} = 0\}. \end{aligned} \quad (4.8)$$

We are interested in the massless matter spectrum induced by the gauge configuration encoded in the rational equivalence class

$$\alpha_X = w_X \cdot_\pi f \in \text{CH}^2(\hat{Y}_4) \quad \text{for } f \in \text{CH}^1(B_3), \quad (4.9)$$

see the discussion around (3.22). The associated $G_4 = \gamma(\alpha_X) \in H^{2,2}(\hat{Y}_4)$ is the gauge flux associated with $U(1)_X$. Recall that this flux is subject to the quantization condition $G_4 + \frac{1}{2}c_2(\hat{Y}_4) \in H_{\mathbb{Z}}^4(\hat{Y}_4)$ [64]. In the present context, a sufficient condition for this is [12, 13] $\gamma(f) + \frac{1}{2}\gamma(D_{\text{GUT}}) \in H_{\mathbb{Z}}^{1,1}(B_3)$. Since $h^1(B_3) = 0$ this is equivalent, at the level of the Picard group, to

$$f + \frac{1}{2}D_{\text{GUT}} \in \text{Pic}(B_3). \quad (4.10)$$

A detailed analysis of the quantization condition for G_4 in more general setups has been performed in [55, 65].

The object $f \in \text{CH}^1(B_3)$ defines an equivalence class of line bundles on B_3 . Let us denote by \mathcal{L} a representative of this line bundle class with $c_1(\mathcal{L}) = \gamma(f)$. Then, according to equ. (3.28),

the massless matter on the respective matter curves is counted by the following cohomology groups:

$$\mathbf{10} \leftrightarrow H^i(C_{10}, \mathcal{L}^{-1}|_{C_{10}} \otimes \sqrt{K_{C_{10}}}), \quad (4.11)$$

$$\bar{\mathbf{5}}_{\mathbf{m}} \leftrightarrow H^i(C_{\bar{\mathbf{5}}_{\mathbf{m}}}, \mathcal{L}^3|_{C_{\bar{\mathbf{5}}_{\mathbf{m}}}} \otimes \sqrt{K_{C_{\bar{\mathbf{5}}_{\mathbf{m}}}}}), \quad (4.12)$$

$$\mathbf{5}_{\mathbf{H}} \leftrightarrow H^i(C_{\mathbf{5}_{\mathbf{H}}}, \mathcal{L}^2|_{C_{\mathbf{5}_{\mathbf{H}}}} \otimes \sqrt{K_{C_{\mathbf{5}_{\mathbf{H}}}}}), \quad (4.13)$$

$$\mathbf{1} \leftrightarrow H^i(C_1, \mathcal{L}^5|_{C_1} \otimes \sqrt{K_{C_1}}). \quad (4.14)$$

The choice of spin structure is, as described around eq. (2.14), the one inherited from the global embedding of the matter curves (4.8) as a complete intersection of 2-cycles with divisor classes D_R^a and D_R^b . For example, if $c_1(K_{B_3}) + D_R^a + D_R^b$ is an even class, it is given, by slight abuse of notation, by $\sqrt{K_{C_R}} = \mathcal{O}_{B_3}(\frac{1}{2}(K_{B_3} + D_R^a + D_R^b))|_{C_R}$ with D_R^a and D_R^b the two divisor classes that define the complete intersection matter curves (4.8).

The computation of the massless spectrum therefore reduces to evaluating the cohomologies of the pullback of a line bundle on B_3 to a codimension-two complete intersection matter curve.

To simplify things further we describe the base B_3 as a hypersurface in a smooth and compact normal toric variety X_Σ ,

$$B_3 = \{p \in X_\Sigma, Q(p) = 0\}, \quad Q \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D_{B_3})), \quad (4.15)$$

such that the divisor class $f \in \text{CH}^1(B_3)$ is itself the pullback of a class $\tilde{f} \in \text{CH}^1(X_\Sigma)$. The same holds for the classes defining the matter curves as complete intersections, which we will denote by the same letter.

Let $\mathcal{O}_{X_\Sigma}(\tilde{f})$ be the line bundle on X_Σ with first Chern class $\gamma(\tilde{f})$. We are then interested in the cohomologies

$$H^i(C_R, L_R|_{C_R}), \quad L_R = \mathcal{O}_{X_\Sigma}\left(q_R \tilde{f} + \frac{1}{2}(K_{X_\Sigma} + D_{B_3} + D_R^a + D_R^b)\right). \quad (4.16)$$

Note that L_R is guaranteed to be integer quantized. In section 4.2 we will describe how to evaluate such cohomology classes, and then apply this general procedure in section 4.3 to a toy model of the type described above that leads to three chiral generations of $SU(5) \times U(1)_X$ matter.

4.2 Pullback cohomologies via *cohomCalg* and the Koszul spectral sequence

According to equ.(4.16) our task is quite generally to compute cohomology groups $H^i(C, \mathcal{L}|_C)$ for a line bundle $\mathcal{L} = \mathcal{O}_{X_\Sigma}(D)$ on a smooth and compact normal toric variety X_Σ and C a smooth subvariety of X_Σ given as a complete intersection. In the case of interest to us, X_Σ is of complex dimension 4 and C is a smooth codimension 3 locus given as a complete intersection

$$C := \{p \in X_\Sigma, \tilde{s}_1(p) = \tilde{s}_2(p) = \tilde{s}_3(p) = 0\} \quad (4.17)$$

with

$$\tilde{s}_i \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(S_i)). \quad (4.18)$$

This problem consists of two parts: First compute the cohomology groups $H^i(X_\Sigma, \mathcal{L})$ and then deduce the desired cohomologies of the pullback bundle $\mathcal{L}|_C$ via the standard technology of Koszul spectral sequences.

On smooth and compact normal toric varieties X_Σ the cohomology classes of holomorphic line bundles can be computed very efficiently with the help of the *cohomCalg* algorithm developed by Blumenhagen and collaborators [44, 45, 46, 47, 48]. Many applications of this algorithm have been described in detail in [48] and [76]. The algorithm [44], implemented in [45], provides a basis of the cohomology groups $H^i(X_\Sigma, \mathcal{L})$ as so-called rationoms. For example, on $X_\Sigma = \mathbb{CP}^3$ the cohomology classes of $\mathcal{L} = \mathcal{O}_{\mathbb{CP}^3}(1)$ are easily computed to be

$$H^0(\mathbb{CP}^3, \mathcal{L}) = \{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3, \alpha_i \in \mathbb{C}\}, \quad H^1(\mathbb{CP}^3, \mathcal{L}) = H^2(\mathbb{CP}^3, \mathcal{L}) = 0 \quad (4.19)$$

Many more examples can be found in [44, 46, 47, 48, 76].

To compute from $H^i(X_\Sigma, \mathcal{L})$ with $\mathcal{L} = \mathcal{O}_{X_\Sigma}(D)$ the cohomologies of the pullback bundle $\mathcal{L}|_C$, it is standard to invoke the sheaf exact Koszul sequence

$$0 \rightarrow \mathcal{L}' \xrightarrow{\alpha} \mathcal{V}_2 \xrightarrow{\beta} \mathcal{V}_1 \xrightarrow{\gamma} \mathcal{L} \xrightarrow{r} \mathcal{L}|_C \rightarrow 0 \quad (4.20)$$

with

$$\mathcal{L}' = \mathcal{O}_{X_\Sigma}(D - S_1 - S_2 - S_3), \quad (4.21)$$

$$\mathcal{V}_2 = \mathcal{O}_{X_\Sigma}(D - S_2 - S_3) \oplus \mathcal{O}_{X_\Sigma}(D - S_1 - S_3) \oplus \mathcal{O}_{X_\Sigma}(D - S_1 - S_2), \quad (4.22)$$

$$\mathcal{V}_1 = \mathcal{O}_{X_\Sigma}(D - S_1) \oplus \mathcal{O}_{X_\Sigma}(D - S_2) \oplus \mathcal{O}_{X_\Sigma}(D - S_3). \quad (4.23)$$

As is well-known, by application of the splitting principle one can replace (4.20) by three short sheaf exact sequences

$$0 \rightarrow \mathcal{L}' \xrightarrow{\alpha} \mathcal{V}_2 \rightarrow \mathcal{I}_1, \quad (4.24)$$

$$0 \rightarrow \mathcal{I}_1 \rightarrow \mathcal{V}_1 \rightarrow \mathcal{I}_2 \rightarrow 0, \quad (4.25)$$

$$0 \rightarrow \mathcal{I}_2 \rightarrow \mathcal{L} \xrightarrow{r} \mathcal{L}|_C \rightarrow 0. \quad (4.26)$$

To each of those there exists an associated long exact sequence in cohomology. Oftentimes exactness of these sequences is sufficient to determine $h^i(C, \mathcal{L}|_C)$ uniquely, and in those cases the Koszul extension of *cohomCalg* [45] can be used to evaluate $h^i(C, \mathcal{L}|_C)$. Unfortunately in many situations these exactness properties are not enough, and to determine $H^i(C, \mathcal{L}|_C)$ from the Koszul spectral sequence one has to explicitly construct the maps in this sequence. This approach goes beyond the Koszul extension of *cohomCalg* and is very similar to the techniques presented in [77].

The mappings in equ. (4.20) are induced from the Koszul complex over \tilde{s}_1, \tilde{s}_2 and \tilde{s}_3 [78, 79]. The appearing sheaf homomorphisms are thus induced by means of the natural restriction of functions from the global-section valued

$$\alpha = \begin{pmatrix} \tilde{s}_1 \\ -\tilde{s}_2 \\ \tilde{s}_3 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & -\tilde{s}_3 & -\tilde{s}_2 \\ -\tilde{s}_3 & 0 & \tilde{s}_1 \\ \tilde{s}_2 & \tilde{s}_1 & 0 \end{pmatrix}, \quad \gamma = (\tilde{s}_1, \tilde{s}_2, \tilde{s}_3). \quad (4.27)$$

\mathcal{L}'	$\check{C}^0(\mathcal{U}, \mathcal{L}') \xrightarrow{\delta} \check{C}^1(\mathcal{U}, \mathcal{L}') \xrightarrow{\delta} \check{C}^2(\mathcal{U}, \mathcal{L}') \xrightarrow{\delta} \check{C}^3(\mathcal{U}, \mathcal{L}') \xrightarrow{\delta} \check{C}^4(\mathcal{U}, \mathcal{L}')$
\downarrow	$\downarrow \alpha^0 \quad \downarrow \alpha^1 \quad \downarrow \alpha^2 \quad \downarrow \alpha^3 \quad \downarrow \alpha^4$
\mathcal{V}_2	$\check{C}^0(\mathcal{U}, \mathcal{V}_2) \xrightarrow{\delta} \check{C}^1(\mathcal{U}, \mathcal{V}_2) \xrightarrow{\delta} \check{C}^2(\mathcal{U}, \mathcal{V}_2) \xrightarrow{\delta} \check{C}^3(\mathcal{U}, \mathcal{V}_2) \xrightarrow{\delta} \check{C}^4(\mathcal{U}, \mathcal{V}_2)$
\downarrow	$\downarrow \beta^0 \quad \downarrow \beta^1 \quad \downarrow \beta^2 \quad \downarrow \beta^3 \quad \downarrow \beta^4$
\mathcal{V}_1	$\check{C}^0(\mathcal{U}, \mathcal{V}_1) \xrightarrow{\delta} \check{C}^1(\mathcal{U}, \mathcal{V}_1) \xrightarrow{\delta} \check{C}^2(\mathcal{U}, \mathcal{V}_1) \xrightarrow{\delta} \check{C}^3(\mathcal{U}, \mathcal{V}_1) \xrightarrow{\delta} \check{C}^4(\mathcal{U}, \mathcal{V}_1)$
\downarrow	$\downarrow \gamma^0 \quad \downarrow \gamma^1 \quad \downarrow \gamma^2 \quad \downarrow \gamma^3 \quad \downarrow \gamma^4$
\mathcal{L}	$\check{C}^0(\mathcal{U}, \mathcal{L}) \xrightarrow{\delta} \check{C}^1(\mathcal{U}, \mathcal{L}) \xrightarrow{\delta} \check{C}^2(\mathcal{U}, \mathcal{L}) \xrightarrow{\delta} \check{C}^3(\mathcal{U}, \mathcal{L}) \xrightarrow{\delta} \check{C}^4(\mathcal{U}, \mathcal{L})$
	$H^0 \quad H^1 \quad H^2 \quad H^3 \quad H^4$

Figure 3: The E_0 -sheet of the Koszul spectral sequence for codimension 3.

\mathcal{L}'	$\check{H}^0(\mathcal{U}, \mathcal{L}')$	$\check{H}^1(\mathcal{U}, \mathcal{L}')$	$\check{H}^2(\mathcal{U}, \mathcal{L}')$	$\check{H}^3(\mathcal{U}, \mathcal{L}')$	$\check{H}^4(\mathcal{U}, \mathcal{L}')$
\downarrow	$\downarrow \tilde{\alpha}^0$	$\downarrow \tilde{\alpha}^1$	$\downarrow \tilde{\alpha}^2$	$\downarrow \tilde{\alpha}^3$	$\downarrow \tilde{\alpha}^4$
\mathcal{V}_2	$\check{H}^0(\mathcal{U}, \mathcal{V}_2)$	$\check{H}^1(\mathcal{U}, \mathcal{V}_2)$	$\check{H}^2(\mathcal{U}, \mathcal{V}_2)$	$\check{H}^3(\mathcal{U}, \mathcal{V}_2)$	$\check{H}^4(\mathcal{U}, \mathcal{V}_2)$
\downarrow	$\downarrow \tilde{\beta}^0$	$\downarrow \tilde{\beta}^1$	$\downarrow \tilde{\beta}^2$	$\downarrow \tilde{\beta}^3$	$\downarrow \tilde{\beta}^4$
\mathcal{V}_1	$\check{H}^0(\mathcal{U}, \mathcal{V}_1)$	$\check{H}^1(\mathcal{U}, \mathcal{V}_1)$	$\check{H}^2(\mathcal{U}, \mathcal{V}_1)$	$\check{H}^3(\mathcal{U}, \mathcal{V}_1)$	$\check{H}^4(\mathcal{U}, \mathcal{V}_1)$
\downarrow	$\downarrow \tilde{\gamma}^0$	$\downarrow \tilde{\gamma}^1$	$\downarrow \tilde{\gamma}^2$	$\downarrow \tilde{\gamma}^3$	$\downarrow \tilde{\gamma}^4$
\mathcal{L}	$\check{H}^0(\mathcal{U}, \mathcal{L})$	$\check{H}^1(\mathcal{U}, \mathcal{L})$	$\check{H}^2(\mathcal{U}, \mathcal{L})$	$\check{H}^3(\mathcal{U}, \mathcal{L})$	$\check{H}^4(\mathcal{U}, \mathcal{L})$
	H^0	H^1	H^2	H^3	H^4

Figure 4: The E_1 -sheet of the Koszul spectral sequence for codimension 3.

From equ. (4.20) the cohomologies of $\mathcal{L}|_C$ can now be computed via the associated Koszul spectral sequence. For the codimension 3 case, this spectral sequence converges on the E_4 -sheet and we display the sheets E_0 to E_4 in Figure 3, Figure 4, Figure 5, Figure 6 and Figure 7. Note that this spectral sequence makes use of the affine open cover \mathcal{U} of X_Σ given by

$$\mathcal{U} = \{U_\sigma, \sigma \in \Sigma_{\max}\}. \quad (4.28)$$

which allows to compute sheaf cohomology from Čech cohomology [79].

The vertical maps in Figure 3 are naturally induced from the sheaf homomorphisms that appear in the Koszul sequence (4.20). Note that the vertical maps preserve closure. The maps in the E_1 -sheet are thus the corresponding maps on the Čech cohomologies. For example we therefore have

$$\tilde{\alpha}^0: \check{H}^0(\mathcal{U}, \mathcal{L}') \rightarrow \check{H}^1(\mathcal{U}, \mathcal{V}_2), [x] \mapsto [\alpha^0(x)]. \quad (4.29)$$

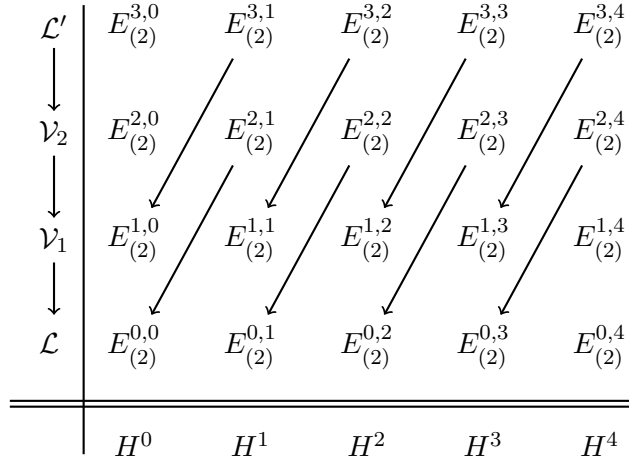


Figure 5: The E_2 -sheet of the Koszul spectral sequence for codimension 3.

The construction of the so-called Knight's moves in the E_2 -sheet, which is displayed in Figure 5, is according to the following principle.

1. Pick $x \in E_{(2)}^{3,1}$.
2. Represent x by $\tilde{x} \in \check{C}^1(\mathcal{U}, \mathcal{L}')$ with $\delta \tilde{x} = 0$ and $\alpha^1(\tilde{x}) = \delta(\tilde{y})$ for a suitable $\tilde{y} \in \check{C}^0(\mathcal{U}, \mathcal{V}_2)$.
3. Set $\tilde{z} := \beta^0(\tilde{y})$ and realise that $\delta(\tilde{z}) = \gamma^0(\tilde{z}) = 0$.
4. Consequently \tilde{z} gives rise to $z \in E_{(2)}^{1,0}$.
5. Define the Knight's move as the map $d_{(2)}^{3,1}: E_{(2)}^{3,1} \rightarrow E_{(2)}^{1,0}$, $x \mapsto z$. This map is independent of the choice of \tilde{y} .

Similarly one constructs the maps in the E_3 -sheet.

From the spaces on the E_4 -sheet one can finally compute the desired pullback cohomologies via [80]

$$H^i(C, \mathcal{L}|_C) = \bigoplus_{j=0}^{\infty} E_{(4)}^{j, i+j}. \quad (4.30)$$

For more details on spectral sequences we refer the interested reader to Appendix C of [79].

Whilst the construction of the maps in the E_2 - and E_3 -sheet along the presented lines is cumbersome, this is in principle the way to construct these maps. Note in particular that in order to follow this principle, one needs knowledge about the E_0 -sheet. This information cannot be obtained from *cohomCalc* as this algorithm immediately computes the cohomology classes in the E_1 -sheet. In order to extract this information one has to follow the classical chamber counting approach [79] which was also followed in [81].

However, sometimes the maps in the sheets E_2 and E_3 can be constructed by simpler means. For example it was used in [82] that the cohomology classes of holomorphic line bundles

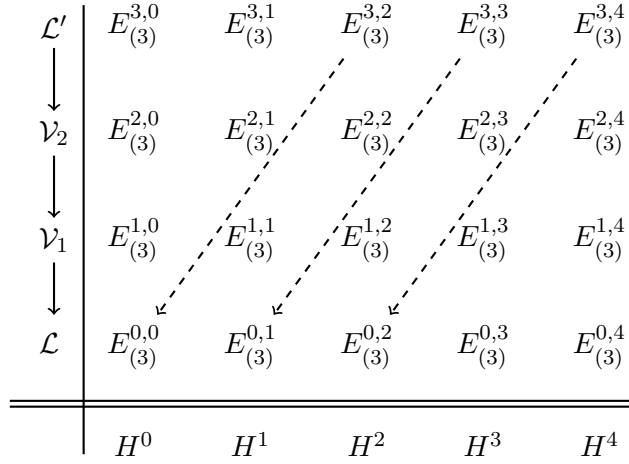


Figure 6: The E_3 -sheet of the Koszul spectral sequence for codimension 3.

on \mathbb{CP}^n are labeled by representations of $U(1) \times U(n)$ [83] and that this extends to direct products of \mathbb{CP}^n s via the Künneth formula. The resulting (anti)-symmetrisation properties of the cohomology classes can then be used to construct the maps in the sheets E_r with $r \geq 2$ more easily.

We have mentioned already that based on *cohomCalc*, the computation of a basis of cohomology for the ambient space cohomologies is possible. Also the maps in the E_1 -sheet are therefore accessible. Our approach is to use the existing C++ implementation of *cohomCalc* whose option *integrated* produces an output styled for use in *Mathematica*. We have then written a *Mathematica* notebook that computes the E_1 -sheet and all maps therein. Details of the maps on the E_1 -sheet are given in appendix A. If the spectral sequence converges on the E_2 -sheet, this is sufficient to compute the cohomologies of the pullback bundle.

Note that our notebook represents the maps in the E_1 -sheet as matrices whose non-zero entries are determined by the complex coefficients of the polynomials \tilde{s}_i . This is also demonstrated in appendix A. Therefore the functionality of the notebook is not limited to generic cases. Rather in leaving these coefficients unconstrained, the notebook provides all the information necessary to study the dependence of the pullback cohomologies on the complex structure of a hypersurface.

4.3 An explicit example

As a concrete example for our base space B_3 , we consider a hypersurface within the toric ambient space $X_\Sigma = \mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ with $K_\Sigma = -3H_1 - 2H_2 - 2H_3$. Here H_i , $i = 1, 2, 3$ represent the respective hyperplane classes on the three factors. In the sequel we will abbreviate a class $aH_1 + bH_2 + cH_3$ as (a, b, c) . The base B_3 is taken to be the vanishing locus of the bundle with class $D_{B_3} = (1, 2, 1)$ such that $K_{B_3} = (-2, 0, -1)|_{B_3}$. The $SU(5)$ divisor is in the class $D_{\text{GUT}} = (1, 0, 1)|_{B_3}$. Since the GUT divisor is not given by a toric divisor, we rewrite the base in terms of a complete intersection such that in the new description it is realised torically [84].

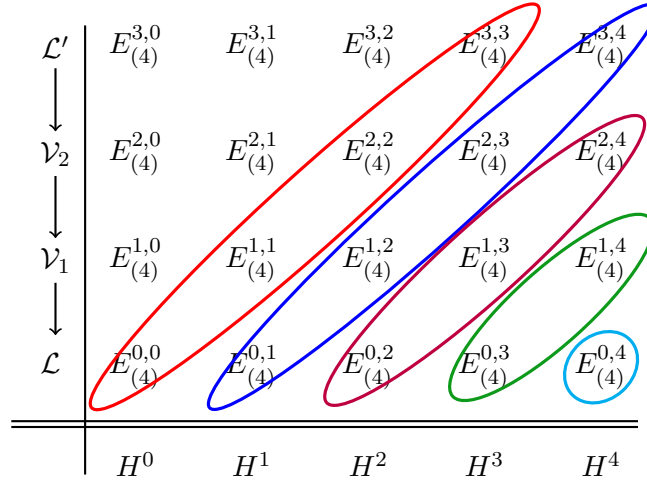


Figure 7: The E_4 -sheet of the Koszul spectral sequence for codimension 3. Note that $H^0(C, \mathcal{L}|_C)$ is obtained by adding the red spaces, $H^1(C, \mathcal{L}|_C)$ is given by the sum of the dark blue spaces and so on.

The variety is embedded in the ambient space with weight matrix

x_1	x_2	x_3	y_1	y_2	z_1	z_2	λ_2	λ_1
1	1	1	0	0	0	0	0	1
0	0	0	1	1	0	0	0	0
0	0	0	0	0	1	1	0	1
0	0	0	0	0	0	0	1	1

(4.31)

and Stanley-Reisner ideal

$$\text{SR-i} : \{y_1 y_2, \lambda_1 \lambda_2, z_1 z_2, x_1 x_2 x_3\}.$$
(4.32)

The base is now the complete intersection of the ‘original’ section

$$P_{(1,2,1,0)}(x_i, y_i, z_i) = 0$$
(4.33)

and a new equation¹⁵

$$\lambda_2 P_{(1,0,1,0)}^{\text{GUT}}(x_i, z_i) = \lambda_1.$$
(4.34)

With this description we can torically construct the elliptically fibred Calabi-Yau 4-fold including the $SU(5)$ gauge theory along the GUT divisor $\{\lambda_1 = 0\}$. Since we are interested in the $U(1)$ -restricted case [32], we take polygon 11 of [85] as the toric ambient space for the fibre. To obtain the non-abelian gauge structure along S_{GUT} , we construct an $SU(5)$ -top [85, 42] such

¹⁵It is important to note that $\lambda_2 = 0$ does not solve (4.34). Therefore, (4.34) defines a section in (4.31), i.e. in the \mathbb{P}^1 bundle over X_Σ .

that $\pi^{-1}(\lambda_1) = e_0 e_1 e_2 e_3 e_4$. The ambient sevenfold reads then as follows:

y	x	z	s	x_1	x_2	x_3	z_1	z_2	y_1	y_2	λ_2	e_0	e_4	e_3	e_2	e_1
0	1	-2	1	-4	0	0	-2	0	0	0	-2	0	1	2	2	1
1	-1	-1	0	-2	0	0	-1	0	0	0	-1	0	1	1	0	0
0	0	0	0	1	0	-1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	-1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	-1	0	0	0	0	0	0
0	0	0	0	0	0	1	0	1	0	0	1	-1	-1	-1	-1	-1
0	0	0	0	0	0	1	1	0	0	0	1	-1	-1	-1	-1	-1

(4.35)

SR-i : $\{y x, y e_2, y e_1, z s, z e_4, z e_3, z e_2, z e_1, y_1 y_2, \lambda_2 e_0, \lambda_2 e_4, \lambda_2 e_3, \lambda_2 e_2,$
 $\lambda_2 e_1, z_1 z_2, s e_0, e_0 e_2, s e_4, s e_2, s e_1, x e_4, e_4 e_2, e_4 e_1, x_1 x_2 x_3,$
 $y e_0 e_3, x e_0 e_3, x e_3 e_1\}$

(4.36)

Together with (4.33), (4.33) and (4.4) this defines the elliptically fibred Calabi-Yau fourfold with the sought-after gauge-symmetries as a complete intersection in a 7-dimensional toric ambient variety. The Euler number of this Calabi-Yau is $\chi(\hat{Y}_4) = 960$.

We now wish to compute the spectrum of $U(1)_X$ charged zero modes that are localised on the curves C_{10} , $C_{\bar{5}_m}$, C_{5_H} and C_1 described generally in (4.8). In the present geometry, they are given as complete intersections on X_Σ of sections with classes

$$C_{10} = (1, 2, 1) \cap (1, 0, 1) \cap (2, 0, 1), \quad C_{\bar{5}_m} = (1, 2, 1) \cap (1, 0, 1) \cap (4, 0, 1), \quad (4.37)$$

$$C_{5_H} = (1, 2, 1) \cap (1, 0, 1) \cap (7, 0, 2), \quad C_1 = (1, 2, 1) \cap (5, 0, 1) \cap (4, 0, 1). \quad (4.38)$$

We consider a gauge configuration of the form (4.9) with $f = \tilde{f}|_{B_3}$ for $\tilde{f} \in \text{CH}^1(X_\Sigma)$. Since X_Σ is a toric variety, rational equivalence and homological equivalence happen to coincide, and \tilde{f} is uniquely characterized by its cohomology class $\gamma(\tilde{f}) \in H^{1,1}(X_\Sigma)$. Whilst we leave it for future work to scan over all admissible choices of classes $\gamma(\tilde{f}) \in H^{1,1}(X_\Sigma)$, we here pick a specific class \tilde{f} which is particularly suited to illustrate the general philosophy. Our choice is

$$\gamma(\tilde{f}) = \frac{1}{2}(-7, 0, 9),$$

which is indeed in agreement with the quantization condition (4.10). Note that this choice is motivated such as to produce three families of chiral matter; the 4-form flux $G_4 = \gamma(\tilde{f})$, however, overshoots the D3-brane tadpole cancellation condition. This has no impact whatsoever on our computation so that our setup suffices as a toy model to illustrate the computation of the massless spectrum in a fully-fledged 4-fold geometry.

According to (4.16) the number of chiral- and anti-chiral modes charged under the $U(1)_X$ symmetry and located on the respective curves are encoded in the cohomologies of the following line bundles:

$$\mathbf{10} \leftrightarrow H^i(C_{10}, \mathcal{L}_1|_{C_{10}}), \quad \mathcal{L}_1 = \mathcal{O}_{X_\Sigma}(4, 0, -4), \quad (4.39)$$

$$\bar{\mathbf{5}}_m \leftrightarrow H^i(C_{\bar{5}_m}, \mathcal{L}_2|_{C_{\bar{5}_m}}), \quad \mathcal{L}_2 = \mathcal{O}_{X_\Sigma}(-9, 0, 14), \quad (4.40)$$

$$\mathbf{5}_H \leftrightarrow H^i(C_{5_H}, \mathcal{L}_3|_{C_{5_H}}), \quad \mathcal{L}_3 = \mathcal{O}_{X_\Sigma}(4, 0, 10), \quad (4.41)$$

$$\mathbf{1} \leftrightarrow H^i(C_1, \mathcal{L}_4|_{C_1}), \quad \mathcal{L}_4 = \mathcal{O}_{X_\Sigma}(-14, 0, 23). \quad (4.42)$$

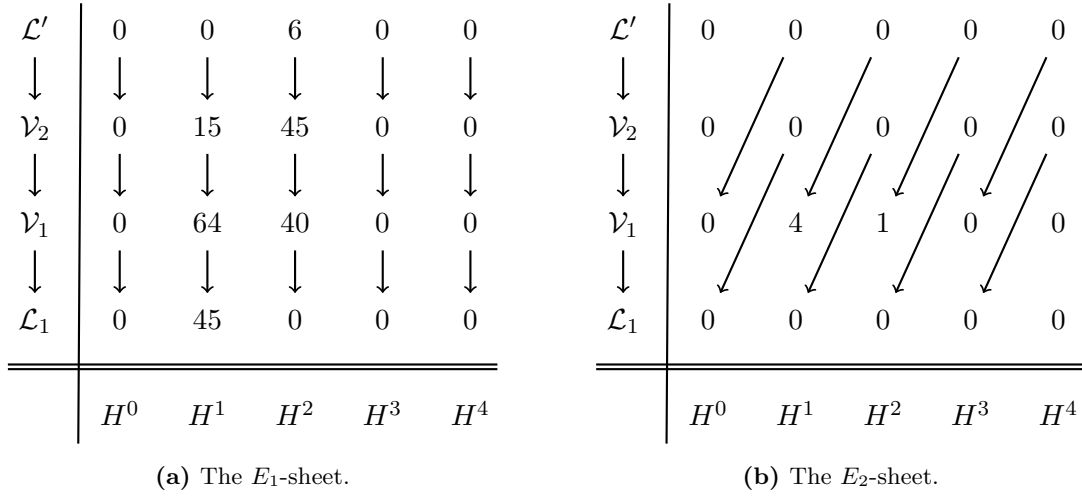


Figure 8: Computation of the cohomologies $h^i(C_{10}, \mathcal{L}_1)$.

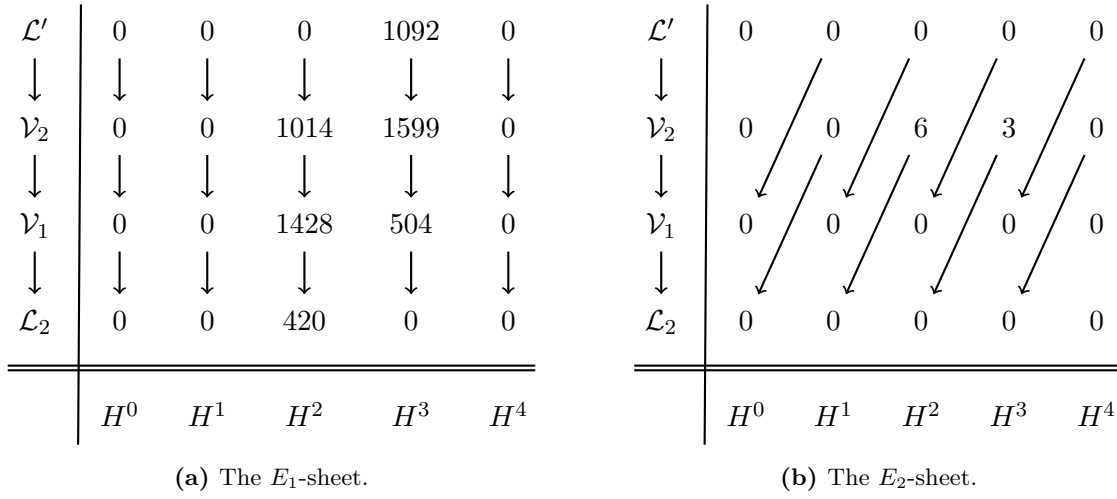


Figure 9: Computation of the cohomologies $h^i(C_{\bar{5}_m}, \mathcal{L}_2)$.

Note that a proper definition of the model requires explicitly specifying the holomorphic section $Q \in H^0(X_\Sigma, B_3)$ defining the base B_3 via (4.15) as well as the Tate polynomials $a_{i,j}$ and the GUT brane normal section w , which enter the definition of the matter curves via (4.8). If we view these sections as polynomials in the homogenous coordinates of X_Σ , the coefficients of the various monomials give an (in general redundant) description of the complex structure moduli of B_3 and the 7-brane moduli. Therefore these coefficients have an influence on the cohomology groups of $L_R|_{C_R}$ appearing in (4.16). Our choice is to pick pseudo-random numbers in the parameter range $(0, 1)$ for all these coefficients. Note in particular that none of the coefficients vanishes. This setup is often referred to as 'maximally generic'.

For such a situation, we have automatised the computation of the E_1 -sheet of the Koszul spectral sequence in a *Mathematica* notebook. The computed E_1 -sheets are displayed in Figure 8, Figure 9, Figure 10 and Figure 11.

In particular this shows that all four spectral sequences converge on the E_2 -sheet. We

\mathcal{L}'	0	0	0	462	0
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
\mathcal{V}_2	0	0	440	530	0
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
\mathcal{V}_1	0	0	465	60	0
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
\mathcal{L}_3	0	0	33	0	0
<hr/>					
	H^0	H^1	H^2	H^3	H^4

(a) The E_1 -sheet.

\mathcal{L}'	0	0	0	0	0
\downarrow	\swarrow	\swarrow	\swarrow	\swarrow	\swarrow
\mathcal{V}_2	0	0	9	8	0
\downarrow	\swarrow	\swarrow	\swarrow	\swarrow	\swarrow
\mathcal{V}_1	0	0	1	0	0
\downarrow	\swarrow	\swarrow	\swarrow	\swarrow	\swarrow
\mathcal{L}_3	0	0	0	0	0
<hr/>					
	H^0	H^1	H^2	H^3	H^4

(b) The E_2 -sheet.

Figure 10: Computation of the cohomologies $h^i(C_{\mathbf{5}_H}, \mathcal{L}_3)$.

\mathcal{L}'	0	0	0	5313	0
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
\mathcal{V}_2	0	0	5082	7128	0
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
\mathcal{V}_1	0	0	6647	2093	0
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
\mathcal{L}_4	0	0	1872	0	0
<hr/>					
	H^0	H^1	H^2	H^3	H^4

(a) The E_1 -sheet.

\mathcal{L}'	0	0	0	278	0
\downarrow	\swarrow	\swarrow	\swarrow	\swarrow	\swarrow
\mathcal{V}_2	0	0	307	0	0
\downarrow	\swarrow	\swarrow	\swarrow	\swarrow	\swarrow
\mathcal{V}_1	0	0	0	0	0
\downarrow	\swarrow	\swarrow	\swarrow	\swarrow	\swarrow
\mathcal{L}_4	0	0	0	0	0
<hr/>					
	H^0	H^1	H^2	H^3	H^4

(b) The E_2 -sheet.

Figure 11: Computation of the cohomologies $h^i(C_1, \mathcal{L}_4)$.

display the so-obtained spectrum of chiral multiplets in table (4.1). We find 4 chiral $\mathcal{N} = 1$ multiplets in the representation $\mathbf{10}_{-1}$ localised on $C_{\mathbf{10}}$. In addition there is one chiral $\mathcal{N} = 1$ multiplet in the $\overline{\mathbf{10}}_1$ representation on $C_{\mathbf{10}}$. Similarly there are 6 multiplets in representation $\overline{\mathbf{5}}_3$ and 3 in the representation $\mathbf{5}_{-3}$ localised on $C_{\overline{\mathbf{5}}_m}$. This gives us 3 chiral families. On $C_{\mathbf{5}_H}$ we have 9 Higgs like doublets.

5 Conclusions and Outlook

In this work we have described a method to extract the number of localised charged massless $\mathcal{N} = 1$ chiral multiplets in an F-theory compactification on an elliptic 4-fold \hat{Y}_4 . Our starting point has been to specify the 3-form data by means of a rational equivalence class of algebraic complex dimension-two cycles $\alpha \in \text{CH}^2(\hat{Y}_4)$. The refined cycle map assigns to α an element of the Deligne cohomology group $H_D^4(\hat{Y}_4, \mathbb{Z}(2))$, which is known [20, 21] to correctly account

curve	$H^0(C, \mathcal{L} _C)$		$H^1(C, \mathcal{L} _C)$	
	$h^0(C, \mathcal{L} _C)$	representation	$h^1(C, \mathcal{L} _C)$	representation
C_{10}	4	$\mathbf{10}_{-1}$	1	$\overline{\mathbf{10}}_{+1}$
$C_{\overline{\mathbf{5}}_m}$	6	$\overline{\mathbf{5}}_3$	3	$\mathbf{5}_{-3}$
$C_{\mathbf{5}_H}$	9	$\mathbf{5}_2$	9	$\overline{\mathbf{5}}_{-2}$
C_1	585	$\mathbf{1}_5$	0	$\overline{\mathbf{1}}_{-5}$

Table 4.1: The chiral $\mathcal{N} = 1$ multiplets in the example presented in subsection 4.3.

for the 3-form data in F/M-theory. Working at the level of Chow groups has the practical advantage that it allows us to very concretely specify the 3-form data. In particular we have described how the intersection product within the Chow ring of \hat{Y}_4 can be used to extract from this data a line bundle L_R on the matter curve C_R which is the natural object counting the number of massless chiral multiplets via $H^i(C_R, L_R \otimes \sqrt{K_{C_R}})$. Our approach is general and does not rely on any local approximation.

Extracting the line bundle L_R on C_R is most immediate if \hat{Y}_4 is embedded into a toric ambient space X_Σ and the 3-form data α arises essentially by pullback from $\text{CH}^2(X_\Sigma)$. In this case, the equivalence of the Chow and the Dolbeault cohomology rings on the toric variety X_Σ implies that we can work within cohomology. We have exemplified this procedure for the 3-form data associated with a $U(1)$ gauge flux and explicitly computed the number of massless charged matter states in an $SU(5) \times U(1)$ toy model.

The obvious next step would be to apply the proposed procedure to a 3-form configuration not of the pullback type focused on in the example of the present paper. This would require carrying out the intersection computation to extract the line bundle L_R directly within the Chow ring of \hat{Y}_4 . An example of such an F-theory 4-form gauge flux has been introduced in [11]. It would be interesting to use our formalism to extend the homological computations of the flux performed in [11] to the level of the underlying 3-form data. Relatedly, the examples considered in the present paper have the simplifying property that they do not induce a Gukow-Vafa-Witten superpotential in the low-energy effective action. In particular, they do not obstruct any of the complex structure moduli of the fourfold. The rich relation between the number of charged localised zero modes and the critical points of this superpotential has been explored in [23] in the context of a representation of the 3-form data via Cheeger-Simons differential forms. A task left for future work is to address configurations with superpotential inducing fluxes in our framework.

Throughout this article we have worked with a smooth resolution \hat{Y}_4 of the singular elliptic fibration. As an advantage, this allows us to study 3-form data within the context of the Deligne cohomology of a *smooth* 4-fold. On the other hand, since the resolution corresponds to moving in the Coulomb branch of the associated 3-dimensional effective action of M-theory compactified on \hat{Y}_4 , the resolved geometry necessarily misses inherently non-abelian aspects of the gauge dynamics. For example the configurations considered here do not include any non-trivial gluing data [86, 29, 30].¹⁶ Interestingly, [24] provides evidence for a definition of the Deligne cohomology of a *singular* elliptic fibration via the Higgs bundle which at least locally

¹⁶See however [87] for the effect of gluing-related monodromies on the resolution geometry.

captures the non-abelian degrees of freedom on the 7-branes [4, 5, 9]. An alternative treatment could be to work not in the resolved, but in the deformed geometry as studied recently in [88].

Despite these slightly more formal considerations to be explored further, we would like to stress that the original motivation for this work was phenomenological: The computation of the number of charged massless vectorlike pairs (in addition to the chiral index) is of obvious importance in model building. The biggest advantage of our approach is therefore its direct applicability to a sufficiently rich set of 3-form data which are employed the explicit construction of realistic string vacua, e.g. of GUT type. Indeed a great deal of work has been invested in the computation of the massless spectrum of realistic heterotic compactifications with cohomological methods - for an incomplete list see e.g. [89, 90, 91, 92, 93, 94, 95] and references therein. In this spirit we look forward to putting Chow groups and Deligne cohomology to work for phenomenological F-theory model building in the near future.

Acknowledgements

We would like to thank Lara Anderson, Hans Jockers, Luca Martucci, Dave Morrison and Eran Palti for useful discussions and correspondence. This work was supported in part by Transregio TR 33 'The Dark Universe' and by the Partnership Mathematics and Physics at Heidelberg University.

A Details on the E_1 -sheet

In this appendix to section 4.2 we explain in more detail the construction of the d_1 -maps appearing on the E_1 -sheet. To state the steps to be performed let us consider the map $\tilde{\alpha}^2: H^2(X_\Sigma, \mathcal{L}') \rightarrow H^2(X_\Sigma, \mathcal{V}_2)$, which we construct by performing the following steps.

1. Determine the rationom basis of $H^2(X_\Sigma, \mathcal{L}')$ and $H^2(X_\Sigma, \mathcal{V}_2)$ based on *cohomCalc*.
2. Multiply the general element x of $H^2(X_\Sigma, \mathcal{L}')$ by the matrix

$$\alpha = \begin{pmatrix} \tilde{s}_1 \\ -\tilde{s}_2 \\ \tilde{s}_3 \end{pmatrix}. \quad (\text{A.1})$$

3. The part of $\alpha \cdot x$ not expressable as a linear combination over \mathbb{C} of the basis of $H^2(X_\Sigma, \mathcal{V}_2)$ is cohomologically zero. Therefore this part can be dropped.
4. Finally express the remaining polynomial vector as a linear combination over \mathbb{C} of the basis of $H^2(X_\Sigma, \mathcal{V}_2)$ and thereby identify the mapping matrix.

Let us exemplify this strategy by considering the hypersurface C in \mathbb{CP}^2 given by

$$C := \{[x_1, x_2, x_3] \in \mathbb{CP}^2, \tilde{s}(x_1, x_2, x_3) := C_1x_1 + C_2x_2 + C_3x_3 = 0\}. \quad (\text{A.2})$$

We now wish to compute the pullback cohomologies of the holomorphic line bundle $\mathcal{O}_{\mathbb{CP}^2}(-3)$ onto this hypersurface C . To this end we make use of the sheaf exact Koszul sequence

$$0 \rightarrow \underbrace{\mathcal{O}_{\mathbb{CP}^2}(-4)}_{=\mathcal{L}'} \xrightarrow{\otimes \tilde{s}} \underbrace{\mathcal{O}_{\mathbb{CP}^2}(-3)}_{=\mathcal{L}} \xrightarrow{r} \mathcal{O}_{\mathbb{CP}^2}(-3)|_C \rightarrow 0. \quad (\text{A.3})$$

$$\begin{array}{c|ccc}
\mathcal{L}' & 0 & 0 & P_1 \\
\downarrow & \downarrow \tilde{\alpha}^0 & \downarrow \tilde{\alpha}^1 & \downarrow \tilde{\alpha}^2 \\
\mathcal{L} & 0 & 0 & P_2 \\
\hline
& H^0 & H^1 & H^2
\end{array}$$

Figure 12: The E_1 -sheet of the Koszul spectral sequence for the hypersurface example.

and display the E_1 -sheet of the associated Koszul spectral sequence in Figure 12. From this picture it is immediately clear that the spectral sequence does converge on the E_2 -sheet and that we need only knowledge about the map

$$\tilde{\alpha}^2: P_1 \rightarrow P_2 \quad (\text{A.4})$$

in order to compute the pullback cohomologies. The spaces P_1 and P_2 are computed based on *cohomCalc* to be

$$P_1 = \left\{ A_1 \left[\frac{1}{x_1 x_2 x_3^2} \right] + A_2 \left[\frac{1}{x_1 x_2^2 x_3} \right] + A_3 \left[\frac{1}{x_1^2 x_2 x_3} \right], A_i \in \mathbb{C} \right\} \cong \mathbb{C}^3, \quad (\text{A.5})$$

$$P_2 = \left\{ B \left[\frac{1}{x_1 x_2 x_3} \right], B \in \mathbb{C} \right\} \cong \mathbb{C}, \quad (\text{A.6})$$

where $[\cdot]$ represents an equivalence class with respect to the Čech complex. On the level of Čech complexes we have the map

$$\alpha^2: \check{C}^2(\mathcal{U}, \mathcal{L}') \rightarrow \check{C}^2(\mathcal{U}, \mathcal{L}), \quad x \mapsto \tilde{s} \cdot x. \quad (\text{A.7})$$

From the commutativity of the E_0 -sheet it follows

$$\alpha^3 \circ \delta(x) = \delta \circ \alpha^2(x). \quad (\text{A.8})$$

Hence if x is closed, so is $\alpha^2(x)$. This enables us to consider the natural map on the Čech cohomologies given by

$$\tilde{\alpha}^2: \check{H}^2(\mathcal{U}, \mathcal{L}') \rightarrow \check{H}^2(\mathcal{U}, \mathcal{L}), \quad [x] \mapsto [\alpha^2(x)] = [\tilde{s} \cdot x]. \quad (\text{A.9})$$

In particular we find for $x := \left[\frac{A_1}{x_1 x_2 x_3^2} + \frac{A_2}{x_1 x_2^2 x_3} + \frac{A_3}{x_1^2 x_2 x_3} \right]$ that $\tilde{\alpha}^2(x) = [M_1] + [M_2]$ where

$$M_1 = \frac{A_1 C_3}{x_1 x_2 x_3} + \frac{A_2 C_2}{x_1 x_2 x_3} + \frac{A_3 C_1}{x_1 x_2 x_3}, \quad (\text{A.10})$$

$$M_2 = \frac{A_2 C_3}{x_1 x_2^2} + \frac{A_3 C_3}{x_1^2 x_2} + \frac{A_1 C_2}{x_1 x_3^2} + \frac{A_1 C_1}{x_2 x_3^2} + \frac{A_3 C_2}{x_1^2 x_3} + \frac{A_2 C_1}{x_2^2 x_3}. \quad (\text{A.11})$$

The crucial observation is that all rationoms in M_2 lie in the image of the Čech coboundary

$$\delta: \check{C}^1(\mathcal{U}, \mathcal{L}) \rightarrow \check{C}^2(\mathcal{U}, \mathcal{L}). \quad (\text{A.12})$$

This can be verified explicitly by computing the Čech cochains as described in [79] (see also [81]). Therefore we ignore the rationoms in M_2 and find

$$\tilde{\alpha}^2(x) = A_1 C_3 \left[\frac{1}{x_1 x_2 x_3} \right] + A_2 C_2 \left[\frac{1}{x_1 x_2 x_3} \right] + A_3 C_1 \left[\frac{1}{x_1 x_2 x_3} \right]. \quad (\text{A.13})$$

From this we can represent the map $\tilde{\alpha}^2$ by the matrix $M_{\tilde{\alpha}^2} = (C_3, C_2, C_1)$. In conclusion we thus find

$$H^0(C, \mathcal{L}|_C) = 0, \quad H^1(C, \mathcal{L}|_C) = \ker(\tilde{\alpha}^2), \quad H^2(C, \mathcal{L}|_C) = \text{coker}(\tilde{\alpha}^2). \quad (\text{A.14})$$

Note that for equ. (A.3) to be sheaf exact, the section \tilde{s} must be non-trivial so that \tilde{s} indeed cuts out a hypersurface. Consequently at least one of the three parameters C_1, C_2, C_3 must be non-zero. In the case at hand this suffices to conclude $\text{im}(\tilde{\alpha}^2) \cong \mathbb{C}$ and therefore

$$h^0(C, \mathcal{L}|_C) = 0, \quad h^1(C, \mathcal{L}|_C) = 2, \quad h^2(C, \mathcal{L}|_C) = 0. \quad (\text{A.15})$$

The vanishing of $h^2(C, \mathcal{L}|_C)$ is expected from the finiteness theorem, whilst the dimensions of h^0 and h^1 are easily confirmed with the Koszul extension of *cohomCalg* [45].

This method is easily generalised e.g. to the ambient space $X_\Sigma = \mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ considered in the example presented in section 4.3.

B Deligne cohomology from the Deligne-Beilinson complex

Recall that we introduced Deligne cohomology in section 2.2 as a generalization of the exact sequence $0 \rightarrow J^1(X) \rightarrow H^1(X, \mathcal{O}_X^*) \rightarrow H_{\mathbb{Z}}^{1,1}(X) \rightarrow 0$ to

$$0 \rightarrow J^p(X) \rightarrow H_D^{2p}(X, \mathbb{Z}(p)) \xrightarrow{\hat{c}_p} H_{\mathbb{Z}}^{p,p}(X) \rightarrow 0, \quad (\text{B.1})$$

that is as an extension of the group of Hodge cycles $H_{\mathbb{Z}}^{p,p}(X)$ by the intermediate Jacobian $J^p(X)$. In this appendix we will review how to arrive at a definition of Deligne Cohomology by thinking about how that short sequence might arise.

Recall that on a Kähler manifold X , there is a Hodge decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}) \quad (\text{B.2})$$

which allows us to define a filtration

$$F^p H^k(X) = \bigoplus_{p' \geq p} H^{p', k-p'} = H^{k,0} \oplus H^{k-1,1} \oplus \dots \oplus H^{p,k-p}. \quad (\text{B.3})$$

The p -th intermediate Jacobian of X is defined to be

$$J^p(X) = \frac{H^{2p-1}(X, \mathbb{C})}{F^p H^{2p-1}(X, \mathbb{C}) + H^{2p-1}(X, \mathbb{Z})}. \quad (\text{B.4})$$

For $p = 1$ this is

$$J^1(X) = \frac{H^1(X, \mathbb{C})}{H^{1,0}(X, \mathbb{C}) + H^1(X, \mathbb{Z})} = H^{0,1}(X, \mathbb{C}) / H^1(X, \mathbb{Z}), \quad (\text{B.5})$$

so it generalizes the classical notion of the Jacobian.

Now note that the integral Hodge cycles can be described as a kernel

$$H_{\mathbb{Z}}^{p,p}(X) = \text{Ker}(H^{2p}(X, \mathbb{Z}) \rightarrow H^{2p}(X, \mathbb{C})/F^p H^{2p}(X, \mathbb{C}))$$

and likewise the p -th intermediate Jacobian is a cokernel

$$J^p(X) = \text{Coker}(H^{2p-1}(X, \mathbb{Z}) \rightarrow H^{2p-1}(X, \mathbb{C})/F^p H^{2p-1}(X, \mathbb{C})),$$

so the exact sequence (B.1) could be deduced from the long exact sequence:

$$\cdots \rightarrow H_D^k(X, \mathbb{Z}(p)) \rightarrow H^k(X, \mathbb{Z}) \rightarrow H^k(X, \mathbb{C})/F^p H^k(X, \mathbb{C}) \rightarrow H_D^{k+1}(X, \mathbb{Z}(p)) \rightarrow \cdots \quad (\text{B.6})$$

This suggests reexpressing $H^k(X, \mathbb{C})/F^p H^k(X, \mathbb{C})$. In order to do this we will need the notion of *hypercohomology of a complex*.

The *holomorphic deRham complex* is the complex of sheaves of holomorphic differential forms Ω_X^\bullet on X ,

$$0 \rightarrow \mathcal{O}_X \rightarrow \Omega_X^1 \rightarrow \cdots \rightarrow \Omega_X^n \rightarrow 0. \quad (\text{B.7})$$

Its *hypercohomology* $\mathbb{H}^k(\Omega_X^\bullet) = H^k(X, \mathbb{C})$ is just ordinary cohomology with complex coefficients by the Dolbeault lemma¹⁷. Moreover it is possible to show that the hypercohomology $\mathbb{H}^k(X, \Omega_X^{\geq p})$ of the truncated holomorphic deRahm complex $\Omega_X^{\geq p} = \Omega_X^p \rightarrow \cdots \rightarrow \Omega_X^n$ is isomorphic to $F^p H^k(X, \mathbb{C})$. So by the short exact sequence of complexes

$$0 \rightarrow \Omega_X^{\geq p} \rightarrow \Omega_X^\bullet \rightarrow \Omega_X^{\leq p-1} \rightarrow 0 \quad (\text{B.8})$$

the hypercohomology of the truncated deRahm complex $\Omega^{\leq p-1}(X)$ is

$$\mathbb{H}^k(\Omega^{\leq p-1}(X)) = H^k(X, \mathbb{C})/F^p H^k(X, \mathbb{C}).$$

Deligne had the insight to consider the following complex, now called the *Deligne-Beilinson complex*,

$$\mathbb{Z}(p)_D = 0 \rightarrow \mathbb{Z} \xrightarrow{(2\pi i)^p} \mathcal{O}_X \rightarrow \Omega_X^1 \rightarrow \cdots \rightarrow \Omega_X^{p-1}. \quad (\text{B.9})$$

By construction, there is a morphism from the truncated holomorphic deRahm complex $\Omega_X^{\leq p-1}[1]$ with degree shifted by one to it, as well as a morphism to \mathbb{Z} , viewed as a complex with one entry. In other words there is a short exact sequence of complexes

$$0 \rightarrow \Omega_X^{\leq p-1}[1] \rightarrow \mathbb{Z}_D(p) \rightarrow \mathbb{Z} \rightarrow 0. \quad (\text{B.10})$$

The crucial insight is that the induced long exact sequence in hypercohomology is precisely the exact sequence (B.1). That is if we define Deligne cohomology to be the hypercohomology of the *Deligne complex* $\mathbb{Z}_D^\bullet(p)$,

$$H_D^k(X, \mathbb{Z}(p)) = \mathbb{H}^k(\mathbb{Z}_D^\bullet(p)), \quad (\text{B.11})$$

then the induced long exact sequence reads

$$\cdots \rightarrow H_D^k(X, \mathbb{Z}(p)) \rightarrow H^k(X, \mathbb{Z}) \rightarrow H^k(X, \mathbb{C})/F^p H^k(X, \mathbb{C}) \rightarrow H_D^{k+1}(X, \mathbb{Z}(p)) \rightarrow \cdots \quad (\text{B.12})$$

¹⁷In the same way that the hypercohomology of the deRahm complex $0 \rightarrow \mathcal{C}_X^\infty \rightarrow \mathcal{A}_X^1 \rightarrow \cdots$ is just ordinary cohomology with real coefficients, by the Poincaré lemma.

There is another way to arrive at the definition of Deligne cohomology. Consider the two embeddings $\alpha_1: \mathbb{Z} \rightarrow \Omega_X^\bullet$ and $\alpha_2: \Omega_X^{\geq p} \rightarrow \Omega_X^\bullet$ in the holomorphic deRahm complex, where α_1 is $(2\pi i)^p$ times the natural inclusion of \mathbb{Z} in \mathcal{O}_X and α_2 is, up to sign, the natural inclusion. Form the cone of the morphism of complexes of sheaves over X :

$$\mathbb{Z} \oplus \Omega_X^{\geq p} \xrightarrow{\alpha_1 - \alpha_2} \Omega_X^\bullet. \quad (\text{B.13})$$

It can be shown that this cone is quasiisomorphic to the Deligne complex (B.9) defined above [22] and therefore their hypercohomologies coincide.

References

- [1] C. Vafa, *Evidence for F-Theory*, *Nucl. Phys.* **B469** (1996) 403–418, [[hep-th/9602022](#)].
- [2] D. R. Morrison and C. Vafa, *Compactifications of F theory on Calabi-Yau threefolds. 1*, *Nucl.Phys.* **B473** (1996) 74–92, [[hep-th/9602114](#)].
- [3] D. R. Morrison and C. Vafa, *Compactifications of F theory on Calabi-Yau threefolds. 2.*, *Nucl.Phys.* **B476** (1996) 437–469, [[hep-th/9603161](#)].
- [4] R. Donagi and M. Wijnholt, *Model Building with F-Theory*, *Adv.Theor.Math.Phys.* **15** (2011) 1237–1318, [[0802.2969](#)].
- [5] C. Beasley, J. J. Heckman and C. Vafa, *GUTs and Exceptional Branes in F-theory - I*, *JHEP* **0901** (2009) 058, [[0802.3391](#)].
- [6] C. Beasley, J. J. Heckman and C. Vafa, *GUTs and Exceptional Branes in F-theory - II: Experimental Predictions*, *JHEP* **01** (2009) 059, [[0806.0102](#)].
- [7] R. Donagi and M. Wijnholt, *Breaking GUT Groups in F-Theory*, *Adv.Theor.Math.Phys.* **15** (2011) 1523–1604, [[0808.2223](#)].
- [8] S. H. Katz and C. Vafa, *Matter from geometry*, *Nucl.Phys.* **B497** (1997) 146–154, [[hep-th/9606086](#)].
- [9] R. Donagi and M. Wijnholt, *Higgs Bundles and UV Completion in F-Theory* [[0904.1218](#)].
- [10] J. Marsano and S. Schafer-Nameki, *Yukawas, G-flux, and Spectral Covers from Resolved Calabi-Yau's*, *JHEP* **1111** (2011) 098, [[1108.1794](#)].
- [11] A. P. Braun, A. Collinucci and R. Valandro, *G-flux in F-theory and algebraic cycles*, *Nucl. Phys.* **B856** (2012) 129–179, [[1107.5337](#)].
- [12] S. Krause, C. Mayrhofer and T. Weigand, *G_4 flux, chiral matter and singularity resolution in F-theory compactifications*, *Nucl.Phys.* **B858** (2012) 1–47, [[1109.3454](#)].
- [13] S. Krause, C. Mayrhofer and T. Weigand, *Gauge Fluxes in F-theory and Type IIB Orientifolds*, *JHEP* **1208** (2012) 119, [[1202.3138](#)].
- [14] A. Clinger, R. Donagi and M. Wijnholt, *The Sen Limit* [[1212.4505](#)].
- [15] T. W. Grimm and H. Hayashi, *F-theory fluxes, Chirality and Chern-Simons theories*, *JHEP* **1203** (2012) 027, [[1111.1232](#)].
- [16] E. Diaconescu, G. W. Moore and D. S. Freed, *The M theory three form and $E(8)$ gauge theory* [[hep-th/0312069](#)].
- [17] D. S. Freed and G. W. Moore, *Setting the quantum integrand of M-theory*, *Commun.Math.Phys.* **263** (2006) 89–132, [[hep-th/0409135](#)].
- [18] G. W. Moore, *Anomalies, Gauss laws, and Page charges in M-theory*, *Comptes Rendus Physique* **6** (2005) 251–259, [[hep-th/0409158](#)].

- [19] J. Cheeger and J. Simons, *Differential characters and geometric invariants*, *Lecture Notes in Math.* **1167** (1985) 50–80.
- [20] G. Curio and R. Y. Donagi, *Moduli in $N=1$ heterotic / F theory duality*, *Nucl.Phys.* **B518** (1998) 603–631, [[hep-th/9801057](#)].
- [21] R. Donagi, *Heterotic / F theory duality: ICMP lecture* [[hep-th/9802093](#)].
- [22] H. Esnault and E. Viehweg, *Beilinson’s conjectures on special values of L -functions*, chapter Deligne-Beilinson cohomology, Waltham: Academic Press, 1988.
- [23] K. Intriligator, H. Jockers, P. Mayr, D. R. Morrison and M. R. Plesser, *Conifold Transitions in M -theory on Calabi-Yau Fourfolds with Background Fluxes* [[1203.6662](#)].
- [24] L. B. Anderson, J. J. Heckman and S. Katz, *T -Branes and Geometry* [[1310.1931](#)].
- [25] M. Green, J. Murre and C. Voisin, *Algebraic Cycles and Hodge Theory*, Springer, 1994.
- [26] D. Eisenbud and J. Harris, *3264 & All that, Intersection Theory and Algebraic Geometry*, 2013.
- [27] W. Fulton, *Intersection Theory*, Princeton University Press 1993 .
- [28] S. H. Katz and E. Sharpe, *D -branes, open string vertex operators, and Ext groups*, *Adv.Theor.Math.Phys.* **6** (2003) 979–1030, [[hep-th/0208104](#)].
- [29] R. Donagi and M. Wijnholt, *Gluing Branes, I*, *JHEP* **1305** (2013) 068, [[1104.2610](#)].
- [30] R. Donagi and M. Wijnholt, *Gluing Branes II: Flavour Physics and String Duality*, *JHEP* **1305** (2013) 092, [[1112.4854](#)].
- [31] J. Marsano, H. Clemens, T. Pantev, S. Raby and H.-H. Tseng, *A Global $SU(5)$ F -theory model with Wilson line breaking*, *JHEP* **1301** (2013) 150, [[1206.6132](#)].
- [32] T. W. Grimm and T. Weigand, *On Abelian Gauge Symmetries and Proton Decay in Global F -theory GUTs*, *Phys.Rev.* **D82** (2010) 086009, [[1006.0226](#)].
- [33] T. W. Grimm and H. Hayashi, *F -theory fluxes, Chirality and Chern-Simons theories*, *JHEP* **1203** (2012) 027, [[1111.1232](#)].
- [34] D. R. Morrison and D. S. Park, *F -Theory and the Mordell-Weil Group of Elliptically-Fibered Calabi-Yau Threefolds*, *JHEP* **1210** (2012) 128, [[1208.2695](#)].
- [35] M. Cvetič, T. W. Grimm and D. Klevers, *Anomaly Cancellation And Abelian Gauge Symmetries In F -theory*, *JHEP* **1302** (2013) 101, [[1210.6034](#)].
- [36] C. Mayrhofer, E. Palti and T. Weigand, *$U(1)$ symmetries in F -theory GUTs with multiple sections*, *JHEP* **1303** (2013) 098, [[1211.6742](#)].
- [37] V. Braun, T. W. Grimm and J. Keitel, *New Global F -theory GUTs with $U(1)$ symmetries*, *JHEP* **1309** (2013) 154, [[1302.1854](#)].
- [38] J. Borchmann, C. Mayrhofer, E. Palti and T. Weigand, *Elliptic fibrations for $SU(5) \times U(1)$ F -theory vacua* [[1303.5054](#)].

- [39] M. Cvetič, D. Klevers and H. Piragua, *F-Theory Compactifications with Multiple $U(1)$ -Factors: Constructing Elliptic Fibrations with Rational Sections*, *JHEP* **1306** (2013) 067, [1303.6970].
- [40] V. Braun, T. W. Grimm and J. Keitel, *Geometric Engineering in Toric F-Theory and GUTs with $U(1)$ Gauge Factors*, *JHEP* **1312** (2013) 069, [1306.0577].
- [41] M. Cvetič, A. Grassi, D. Klevers and H. Piragua, *Chiral Four-Dimensional F-Theory Compactifications With $SU(5)$ and Multiple $U(1)$ -Factors* [1306.3987].
- [42] J. Borchmann, C. Mayrhofer, E. Palti and T. Weigand, *$SU(5)$ Tops with Multiple $U(1)$ s in F-theory* [1307.2902].
- [43] M. Cvetič, D. Klevers, H. Piragua and P. Song, *Elliptic Fibrations with Rank Three Mordell-Weil Group: F-theory with $U(1) \times U(1) \times U(1)$ Gauge Symmetry* [1310.0463].
- [44] R. Blumenhagen, B. Jurke, T. Rahn and H. Roschy, *Cohomology of Line Bundles: A Computational Algorithm*, *J. Math. Phys.* **51** (2010) 103525, [1003.5217].
- [45] *cohomCalc package*, Download link, 2010, URL <http://wwwth.mppmu.mpg.de/members/blumenha/cohom> high-performance line bundle cohomology computation based on [44].
- [46] S.-Y. Jow, *Cohomology of toric line bundles via simplicial Alexander duality*, *Journal of Mathematical Physics* **52**, no. 3 (2011) 033506, [1006.0780].
- [47] T. Rahn and H. Roschy, *Cohomology of Line Bundles: Proof of the Algorithm*, *J.Math.Phys.* **51** (2010) 103520, [1006.2392].
- [48] R. Blumenhagen, B. Jurke, T. Rahn and H. Roschy, *Cohomology of Line Bundles: Applications*, *J.Math.Phys.* **53** (2012) 012302, [1010.3717].
- [49] P. S. Aspinwall, *D-branes on Calabi-Yau manifolds* pp. 1–152, [hep-th/0403166].
- [50] J. Distler and B. R. Greene, *Aspects of $(2,0)$ String Compactifications*, *Nucl.Phys.* **B304** (1988) 1.
- [51] D. S. Freed and E. Witten, *Anomalies in string theory with D-branes* [hep-th/9907189].
- [52] R. Blumenhagen, T. W. Grimm, B. Jurke and T. Weigand, *Global F-theory GUTs*, *Nucl. Phys.* **B829** (2010) 325–369, [0908.1784].
- [53] T. W. Grimm, S. Krause and T. Weigand, *F-Theory GUT Vacua on Compact Calabi-Yau Fourfolds*, *JHEP* **07** (2010) 037, [0912.3524].
- [54] C.-M. Chen, J. Knapp, M. Kreuzer and C. Mayrhofer, *Global $SO(10)$ F-theory GUTs*, *JHEP* **1010** (2010) 057, [1005.5735].
- [55] A. Collinucci and R. Savelli, *On Flux Quantization in F-Theory*, *JHEP* **1202** (2012) 015, [1011.6388].
- [56] J. Knapp, M. Kreuzer, C. Mayrhofer and N.-O. Walliser, *Toric Construction of Global F-Theory GUTs*, *JHEP* **1103** (2011) 138, [1101.4908].

- [57] M. Esole and S.-T. Yau, *Small resolutions of $SU(5)$ -models in F -theory* [1107.0733].
- [58] C. Lawrie and S. Schfer-Nameki, *The Tate Form on Steroids: Resolution and Higher Codimension Fibers*, *JHEP* **1304** (2013) 061, [1212.2949].
- [59] H. Hayashi, C. Lawrie and S. Schafer-Nameki, *Phases, Flops and F -theory: $SU(5)$ Gauge Theories*, *JHEP* **1310** (2013) 046, [1304.1678].
- [60] H. Hayashi, C. Lawrie, D. R. Morrison and S. Schafer-Nameki, *Box Graphs and Singular Fibers* [1402.2653].
- [61] K. Becker and M. Becker, *M theory on eight manifolds*, *Nucl.Phys.* **B477** (1996) 155–167, [hep-th/9605053].
- [62] S. Sethi, C. Vafa and E. Witten, *Constraints on low dimensional string compactifications*, *Nucl.Phys.* **B480** (1996) 213–224, [hep-th/9606122].
- [63] K. Dasgupta, G. Rajesh and S. Sethi, *M theory, orientifolds and G - flux*, *JHEP* **9908** (1999) 023, [hep-th/9908088].
- [64] E. Witten, *On flux quantization in M theory and the effective action*, *J.Geom.Phys.* **22** (1997) 1–13, [hep-th/9609122].
- [65] A. Collinucci and R. Savelli, *On Flux Quantization in F-Theory II: Unitary and Symplectic Gauge Groups*, *JHEP* **1208** (2012) 094, [1203.4542].
- [66] K. Intriligator, H. Jockers, P. Mayr, D. R. Morrison and M. R. Plesser, *Conifold Transitions in M-theory on Calabi-Yau Fourfolds with Background Fluxes* [1203.6662].
- [67] C. Voisin, *Hodge Theory and Complex Algebraic Geometry I*, Cambridge University Press 2002 .
- [68] G. E. Bredon, *Topology and Geometry*, Springer, 1997.
- [69] R. Donagi and M. Wijnholt, *Higgs Bundles and UV Completion in F-Theory* [0904.1218].
- [70] W. Fulton, *Introduction to Toric Varieties*, Springer 1997 .
- [71] A. Klemm, P. Mayr and C. Vafa, *BPS states of exceptional noncritical strings* [hep-th/9607139].
- [72] G. Aldazabal, A. Font, L. E. Ibanez and A. Uranga, *New branches of string compactifications and their F theory duals*, *Nucl.Phys.* **B492** (1997) 119–151, [hep-th/9607121].
- [73] P. Candelas, E. Perevalov and G. Rajesh, *Comments on A, B, C chains of heterotic and type II vacua*, *Nucl.Phys.* **B502** (1997) 594–612, [hep-th/9703148].
- [74] P. Berglund, A. Klemm, P. Mayr and S. Theisen, *On type IIB vacua with varying coupling constant*, *Nucl.Phys.* **B558** (1999) 178–204, [hep-th/9805189].
- [75] T. Shioda, *On the Mordell-Weil lattices*, *Comment. Math. Univ. St. Pauli* **39** (1990) 211–240.

- [76] T. Rahn, *Heterotic Target Space Dualities with Line Bundle Cohomology*, Ph.D. thesis, Ludwig-Maximilians-Universitt Mnchen, 2012, URL http://edoc.ub.uni-muenchen.de/14344/2/Rahn_Thorsten.pdf.
- [77] L. B. Anderson, J. Gray, A. Lukas and B. Ovrut, *Vacuum Varieties, Holomorphic Bundles and Complex Structure Stabilization in Heterotic Theories*, *JHEP* **1307** (2013) 017, [[1304.2704](#)].
- [78] D. Eisenbud, *Commutative Algebra: With a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, Springer, 1995, ISBN 9780387942698, URL http://books.google.de/books?id=Fm_yPgZBucMC.
- [79] D. Cox, J. Little and H. Schenck, *Toric Varieties*, Graduate studies in mathematics, American Mathematical Society, 2011, ISBN 9780821884263, URL <http://books.google.de/books?id=eXLGwYD4pmAC>.
- [80] J. Distler and B. Greene, *Aspects of $(2, 0)$ string compactifications*, *Nuclear Physics B* **304** (1988) 1–62.
- [81] M. Cvetič, I. Garcia-Etxebarria and J. Halverson, *Global F-theory Models: Instantons and Gauge Dynamics*, *JHEP* **1101** (2011) 073, [[1003.5337](#)].
- [82] L. B. Anderson, *Heterotic and M-theory Compactifications for String Phenomenology* [[0808.3621](#)].
- [83] T. Hübsch, *Calabi-Yau Manifolds: A Bestiary for Physicists*, World Scientific, 1994, ISBN 9789810219277, URL <http://books.google.de/books?id=Z5zSbFktn1EC>.
- [84] A. Collinucci and R. Savelli, *On Flux Quantization in F-Theory*, *JHEP* **1202** (2012) 015, [[1011.6388](#)].
- [85] V. Bouchard and H. Skarke, *Affine Kac-Moody algebras, CHL strings and the classification of tops*, *Adv. Theor. Math. Phys.* **7** (2003) 205–232, [[hep-th/0303218](#)].
- [86] S. Cecotti, C. Cordova, J. J. Heckman and C. Vafa, *T-Branes and Monodromy*, *JHEP* **1107** (2011) 030, [[1010.5780](#)].
- [87] A. P. Braun and T. Watari, *On Singular Fibres in F-Theory*, *JHEP* **1307** (2013) 031, [[1301.5814](#)].
- [88] A. Grassi, J. Halverson and J. L. Shaneson, *Matter From Geometry Without Resolution* [[1306.1832](#)].
- [89] R. Donagi, Y.-H. He, B. A. Ovrut and R. Reinbacher, *The Particle spectrum of heterotic compactifications*, *JHEP* **0412** (2004) 054, [[hep-th/0405014](#)].
- [90] R. Donagi, Y.-H. He, B. A. Ovrut and R. Reinbacher, *The Spectra of heterotic standard model vacua*, *JHEP* **0506** (2005) 070, [[hep-th/0411156](#)].
- [91] V. Bouchard and R. Donagi, *An $SU(5)$ heterotic standard model*, *Phys.Lett.* **B633** (2006) 783–791, [[hep-th/0512149](#)].

- [92] V. Braun, Y.-H. He, B. A. Ovrut and T. Pantev, *Vector bundle extensions, sheaf cohomology, and the heterotic standard model*, *Adv.Theor.Math.Phys.* **10** (2006) 525–589, [[hep-th/0505041](#)].
- [93] R. Blumenhagen, S. Moster, R. Reinbacher and T. Weigand, *Massless Spectra of Three Generation $U(N)$ Heterotic String Vacua*, *JHEP* **0705** (2007) 041, [[hep-th/0612039](#)].
- [94] L. B. Anderson, J. Gray, A. Lukas and E. Palti, *Two Hundred Heterotic Standard Models on Smooth Calabi-Yau Threefolds*, *Phys.Rev.* **D84** (2011) 106005, [[1106.4804](#)].
- [95] L. B. Anderson, J. Gray, A. Lukas and E. Palti, *Heterotic Line Bundle Standard Models*, *JHEP* **1206** (2012) 113, [[1202.1757](#)].